Approximate Solution of the Simple Hypersingular Integral Equations with Hilbert Kernel

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***Abstract* -In the present paper, the hypersingular integral operator with Hilbert kernel is approximated by a sequence of operators of the special form, it is proved that, the approximating operators strongly converge to the operator of the special form, for a trigonometric polynomial of degree not higher than n and is given a new method for the approximate solution of hypersingular integral equations of the first kind with Hilbert kernel and as an example is shown its application to a certain problem. The proposed method is straightforward to calculate, making the new approximate method reliable and easy to apply and the obtained numerical results demonstrate the stability and efficiency of the approach.**

Key words -hypersingular integral operator, Hilbert kernel, hypersingular integral equation, best mean-square approximation, convergence

I.INTRODUCTION

Active development of numerical methods for solving hypersingular integral equations is of considerable interest in modern numerical analysis. This is because the fact that hypersingular integral equations have numerous applications in acoustics, aerodynamics, fluid mechanics, electrodynamics, elasticity, fracture mechanics, geophysics and etc. [2,8]. The advantages of this method are as follows: using this method, we obtain an immediate estimate of the convergence rate for functions of the appropriate class, while other methods used earlier to approximate solutions of hypersingular integral equations require the study of the specific properties of the function classes under consideration; moreover, the estimate established in this paper yields more exact results in terms of the convergence rate than the methods used earlier [9-13,18] approximate solutions of the system of linear algebraic equations can be found explicitly (not at isolated points) and, therefore, given a fixed convergence rate, this method requires smaller computational resources.

It is known that, the solution of the inner Neumann problem for Laplace equation reduced to the following hypersingular integral equation

$$(R\varphi)(t)=\frac{1}{4\pi}\int_0^{2\pi}\csc^2\frac{\tau-t}{2}\varphi(\tau)d\tau+$$
$$+\int_0^{2\pi}K(t,\tau)\varphi(\tau)d\tau=f(t)$$

where $t\in T_0=[0;2\pi]$ and $K(t,\tau)=\frac{\partial}{\partial t}F(t,\tau)$ is 2π periodic for all arguments and continuous function.

In the present paper, the operator R is approximated by a sequence of operators of the form

$$(R_n\varphi)(t)=\sum_{k=0}^{2n-1}\alpha_k^{(n)}(t)\varphi\left(t+\frac{\pi k}{n}\right),\quad t\in T_0,\ n\in N,$$

where the functions $\alpha_k^{(n)}(t)$ are continuous functions, $k=0,1,2,\ldots,$ $n\in N$ and expressed in terms of the given functions and is given a new method for the approximate solution of hypersingular integral equations of the first kind.

It should be noted that, for the operators $(R_n\varphi)(t)$, the determination of the inverse operator is equivalent to the study of the equation

$$\sum_{k=0}^{2n-1}\alpha_k^{(n)}(t)\varphi\left(t+\frac{\pi k}{n}\right)=f(t)$$

at the points $t,\ t+\frac{\pi}{n},\ldots,\ t+\frac{\pi(2n-1)}{n}$, because solving the resulting system of linear algebraic equations with respect to

$$\left(\varphi(t),\ \varphi\left(t+\frac{\pi}{n}\right),\ldots,\varphi\left(t+\frac{\pi(2n-1)}{n}\right)\right),$$

we obtain the function $\varphi(t)$.

Note that, for the singular integral operators with Cauchy kernel and Hilbert kernel similar approximations and its application to singular integral equations are given in the papers [1,4-6,13-16].

II. HYPERSINGULAR HILBERT INTEGRAL.

Consider the integral

$$\int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau \,, t \in T_0 \,, \qquad (1)$$

where φ is Lebesgue integrable on T_0 and 2π-periodic function. If we define this integral similar to the Cauchy integral, even if $\varphi \equiv 1$, we get the divergent integral.

Therefore, using the idea of Hadamard [9], we will define the integral (1) as follows:

Definition 1. If a finite limit

$$\lim_{\varepsilon \to 0+} \left(\int_{t-\pi}^{t-\varepsilon} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau + \int_{t+\varepsilon}^{t+\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau - \frac{8\varphi(t)}{\varepsilon} \right)$$

exists, then the value of this limit is referred to as the hypersingular Hilbert integral of the function φ on T_0, and is denoted by $\int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau$.

From the definition 1 it follows that,

$$\int_0^{2\pi} \csc^2 \frac{\tau - t}{2} d\tau == \lim_{\varepsilon \to 0+} \left(4 \cot \frac{\varepsilon}{2} - \frac{8}{\varepsilon} \right) = 0$$

and, therefore,

$$\int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau = \int_0^{2\pi} \csc^2 \frac{\tau - t}{2} [\varphi(\tau) - \varphi(t)] d\tau \,,$$

where the integral standing in the right side is understood in the sense of the Cauchy principal value.

In the paper [2], it is proved that, if $\varphi(\tau) \in H_1(\alpha)$, then there exist the hypersingular Hilbert integral (1) and the following equation holds:

$$\int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau = 2 \int_0^{2\pi} \cot \frac{\tau - t}{2} \varphi'(\tau) d\tau \,, \qquad (2)$$

where the integral standing in the right side is understood in the sense of the Cauchy principal value.

Theorem 1. If the 2π-periodic function φ is absolutely continuous on T_0, then the hypersingular Hilbert integral (1) exists for almost all $t \in T_0$, and the equation (2) holds.

III. APPROXIMATION OF THE HYPERSINGULAR INTEGRAL OPERATORS WITH HILBERT KERNEL.

Let $L_2 = L_2(T_0)$ be the space of the functions square-integrable on T_0 with the norm

$$\|\varphi\|_{L_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} |\varphi(\tau)|^2 d\tau \right)^{\frac{1}{2}} ,$$

and let $W_2^1 = W_2^1(T_0)$ be the space of absolutely continuous functions on T_0, which their derivatives belong on the space L_2, with the norm

$$\|\varphi\|_{W_2^1} = \|\varphi\|_{L_2} + \|\varphi'\|_{L_2} .$$

Since the following hypersingular integral operator with Hilbert kernel

$$(H\varphi)(t) = \frac{1}{4\pi} \int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau \,, \ t \in T_0$$

is bounded from the space W_2^1 into the space L_2, and $\|H\|_{W_2^1 \to L_2} = 1$,[14].

Consider the sequence of operators

$$(H_n\varphi)(t) = \frac{1}{2n} \sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n} \left(\varphi\left(t + \frac{\pi(2k+1)}{n} \right) - \varphi(t) \right),$$

$$t \in T_0 \,, \ n \in N .$$

Let us calculate $H_n\left(e^{imt}\right)$ for any $m \in Z$ (Z is the set of integer real numbers):

$$H_n\left(e^{imt}\right) = \frac{1}{2n} \sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n} \times$$
$$\times \left(\exp\left(imt + \frac{im\pi(2k+1)}{n} \right) - \exp(imt) \right) = \mu_m^{(n)} e^{imt} \,, \qquad (3)$$

where

$$\mu_m^{(n)} = \frac{1}{2n} \sum_{k=0}^{n-1} \csc^2 \frac{\pi(2k+1)}{2n} \left(\exp \frac{im\pi(2k+1)}{n} - 1 \right) .$$

Since

$$\mu_0^{(n)} = 0,$$

$$\mu_{m+1}^{(n)} - \mu_m^{(n)} = \frac{i}{n} \sum_{k=0}^{n-1} \csc \frac{\pi(2k+1)}{2n} \times$$
$$\times \exp \frac{i(m+1/2)\pi(2k+1)}{n} = \lambda_m^{(n)} ,$$

As

$$\lambda_0^{(n)} = -1 ,$$

$$\lambda_{m+1}^{(n)} - \lambda_m^{(n)} = -\frac{2}{n} \sum_{k=0}^{n-1} \exp \frac{i(m+1)\pi(2k+1)}{n} = \delta_m^{(n)} \,,$$

and $\delta_m^{(n)} = -2$ for $m = -1 (\operatorname{mod} 2n)$, $\delta_m^{(n)} = 2$ for $m = n - 1 (\operatorname{mod} 2n)$, $\delta_m^{(n)} = 0$ for $m \neq -1 (\operatorname{mod} n)$, it follows that, $\lambda_m^{(n)} = -1$ for $m = \overline{0, n-1}$, $\lambda_m^{(n)} = 1$ for $m = \overline{n, 2n-1}$, $\lambda_{m\pm 2n}^{(n)} = \lambda_m^{(n)}$ for all $m \in Z$, and, therefore, $\mu_m^{(n)} = -m$ for $m = \overline{0, n}$, $\mu_m^{(n)} = m - 2n$ for $m = \overline{n+1, 2n}$, $\mu_{m\pm 2n}^{(n)} = \mu_m^{(n)}$ for all $m \in Z$.

Suppose that $\varphi(t)=\sum_{k=-\infty}^{+\infty} c_k e^{ikt} \in W_2^1$ .Then, taking (3) into account, we obtain

$$(H_n\varphi)(t)=\sum_{k=-\infty}^{+\infty} c_k \mu_k^{(n)} e^{ikt} .$$

Since the coefficients $\mu_k^{(n)}$ satisfies the inequality $\left|\mu_k^{(n)}\right| \le |k|$ for all $k \in Z$, it follows that

$$\|H_n\varphi\|_{L_2} = \sum_{k=-\infty}^{+\infty} |c_k|^2 \left|\mu_k^{(n)}\right|^2 \le \sum_{k=-\infty}^{+\infty} k^2 |c_k|^2 \le \|\varphi\|_{W_2^1} .$$

This implies the following properties of the operators H_n :

Property 1.[3]The operators H_n , $n=1,2,...$ is bounded from the space W_2^1 into the space L_2 , $\|H_n\|_{W_2^1 \to L_2} \le 1$, and for any trigonometric polynomial $q(t)=\sum_{k=-n}^{n} q_k e^{ikt}$ the following relation holds:

$$(H_n q)(t)=(Hq)(t) .$$

Suppose that $E_n\left(\varphi; W_2^1\right)=\inf_{q \in T_n} \|\varphi(\cdot)-q_n(\cdot)\|_{W_2^1}$ is the best approximation of the function $\varphi \in W_2^1$ by polynomials T_n , where T_n is the set of trigonometric polynomials of the form $\sum_{k=-n}^{n} \alpha_k e^{ikt}$, $\alpha_k \in C$.

Property 2. [3] The sequence of operators $\{H_n\}$ strongly converges to the operator H and, for any $\varphi \in W_2^1$, the following estimate holds:

$$\|H\varphi - H_n\varphi\|_{L_2} \le 2E_n\left(\varphi; W_2^1\right).$$

IV. THE APPROXIMATE SOLUTION OF THE HYPERSINGULAR INTEGRAL EQUATIONS OF THE FIRST KIND

At first consider the simple hypersingular integral equation of the first kind

$$(H\varphi)(t)=f(t),\ t \in T_0 , \tag{4}$$

where $f \in L_2$. Since for any function $\varphi \in W_2^1$ the equation

$$\int_0^{2\pi} (H\varphi)(\tau)d\tau = \int_0^{2\pi} (S\varphi')(\tau)d\tau = \int_0^{2\pi} \varphi'(\tau)d\tau = 0 ,$$

holds, then the equation (4) is solvable only if

$$\int_0^{2\pi} f(\tau)d\tau = 0 , \tag{5}$$

where $(S\varphi)(t) \equiv \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau - t}{2}\varphi(\tau)d\tau$, $t \in T_0$ is singular integral operator with Hilbert kernel. If the condition (5) satisfies, then the equation (4) has infinitely many solutions in the general form

$$\varphi^*(t)=d_0 - \sum_{k \in Z\setminus\{0\}} \frac{c_k(f)}{|k|} e^{ikt} \in W_2^1 , \tag{6}$$

Where $c_k(f)=\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\tau} f(\tau)d\tau$ is Fourier's coefficients of function $f \in L_2$, and d_0 is a constant.Therefore, if we consider the equation

$$(H\varphi)(t)=f(t)-\frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau ,t \in I , \tag{7}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(\tau)d\tau = d_0 , \tag{8}$$

then, it follows that the equation (7)-(8) is unique solvable for any $(f;d_0) \in L_2 \times C$, and the solution of (7)-(8) is the function, defined by (6).

Now consider the following equation

$$(H_n\varphi)(t)=f(t), t \in T_0 , \tag{9}$$

where $f \in L_2$ .Considering the equation (9) at the points $t,\ t+\frac{\pi}{n},...,\ t+\frac{\pi(2n-1)}{n}$ we obtain the following system of linear algebraic equations:

$$(H_n\varphi)\left(t+\frac{\pi k}{n}\right)=f\left(t+\frac{\pi k}{n}\right),\quad k=\overline{0,2n-1}\ ,\ t \in T_0$$

with respect to $\left(\varphi(t),\ \varphi\left(t+\frac{\pi}{n}\right),...,\varphi\left(t+\frac{\pi(2n-1)}{n}\right)\right)$.

Since for any $t \in T_0$ the equation

$$\sum_{k=0}^{2n-1} (H_n\varphi)\left(t+\frac{\pi k}{n}\right)=0 ,$$

holds, then we obtain that, the equation (9) is solvable only if

$$\sum_{k=0}^{2n-1} f\left(t+\frac{\pi k}{n}\right)=0 . \tag{10}$$

If the condition (10) is satisfies, then the equation (9) has infinitely many solutions. And if we consider the equation

$$(H_n\varphi)(t)=f(t)-\frac{1}{2n}\sum_{k=0}^{2n-1} f\left(t+\frac{\pi k}{n}\right),t \in T_0 , \tag{11}$$

$$\frac{1}{2n}\sum_{k=0}^{2n-1} \varphi\left(t+\frac{\pi k}{n}\right)=d_0 , \tag{12}$$

We obtain that, the equations (11)-(12) is unique solvable for any $(f;d_0)\in L_2\times C$ and the solutions of (11)-(12) are the following functions

$$\varphi_n^*(t)=d_0+\sum_{\substack{k=-\infty\\ k\neq 0\ (\mathrm{mod}\ 2n)}}^{+\infty}\frac{c_k(f)}{\mu_k^{(n)}}e^{ikt}\in L_2 .$$

Theorem 2. For any $(f;d_0)\in L_2\times C$ the system of linear algebraic equations (11)-(12) are unique solvable with respect to

$$\left(\varphi(t),\ \varphi\left(t+\frac{\pi}{n}\right),\dots,\varphi\left(t+\frac{\pi(2n-1)}{n}\right)\right)$$

the solutions $\varphi_n^*(t)$ of the equations (11)-(12) converge to the solution $\varphi^*(t)$ of the equation (7)-(8) in the norm of the space L_2 and the following estimate holds:

$$\left\|\varphi_n^*-\varphi^*\right\|_{L_2}\le E_n(f;L_2).$$

Proof. From the equations (7) and (8) it follows that:

$$\left\|\varphi_n^*-\varphi^*\right\|_{L_2}=\left\|\begin{array}{c}\displaystyle\sum_{\substack{|k|>n\\ k\neq 0\ (\mathrm{mod}\ 2n)}}\frac{c_k(f)}{\mu_k^{(n)}}e^{ikt}+\\ \displaystyle+\sum_{|k|>n}\frac{c_k(f)}{|k|}e^{ikt}\end{array}\right\|_{L_2}\le .$$

$$\le\left(\sum_{|k|>n}\left|c_k(f)\right|^2\right)^{\frac{1}{2}}\le E_n(f;L_2)$$

This completes the proof of the theorem.

V. NUMERICAL EXAMPLES.

Example 1. In numerical examples most of the authors consider the continuous and bounded functions. To show the efficiency and accuracy of our methods, we will consider as an example the unbounded function. Consider the following hypersingular integral equation

$$(H\varphi)(t)=\frac{2}{\pi}\ln\left|\tan\frac{t}{2}\right|, \tag{13}$$

with the condition

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi(\tau)d\tau=\frac{\pi}{2}. \tag{14}$$

It is easy to verify that the function $\varphi^*(t)=|t-\pi|$ is a solution of (13) - (14) on the segment $[0,2\pi]$.

Now consider the equation

$$(H_n\varphi)(t)=\frac{2}{\pi}\ln\left|\tan\frac{t}{2}\right|, \tag{15}$$

$$\frac{1}{2n}\sum_{k=0}^{2n-1}\varphi\left(t+\frac{\pi k}{n}\right)=\frac{\pi}{2}. \tag{16}$$

If we study the equation (15) at the points $t+\frac{\pi k}{n}$, $k=\overline{0,2n-2}$, and add the equation (16) we will obtain the system of linear algebraic equations with respect to $\left(\varphi(t),\ \varphi\left(t+\frac{\pi}{n}\right),\dots,\varphi\left(t+\frac{\pi(2n-1)}{n}\right)\right)$.

Fix the point $t_0=\frac{\pi}{2n}$ and solving the system of linear algebraic equations

$$\begin{cases}\displaystyle\frac{1}{2n}\sum_{k=0}^{n-1}\csc^2\frac{\pi(2k+1)}{2n}\left(\varphi\left(\frac{\pi(4k+3)}{2n}\right)-\varphi\left(\frac{\pi}{2n}\right)\right)=\frac{2}{\pi}\ln\left|\tan\frac{\pi}{4n}\right|,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \displaystyle\frac{1}{2n}\sum_{k=0}^{n-1}\csc^2\frac{\pi(2k+1)}{2n}\left(\varphi\left(\frac{\pi(4k-1)}{2n}\right)-\varphi\left(\frac{\pi(4n-3)}{2n}\right)\right)=\frac{2}{\pi}\ln\left|\tan\frac{\pi(4n-3)}{4n}\right|,\\ \displaystyle\frac{1}{2n}\sum_{k=0}^{2n-1}\varphi\left(\frac{\pi(2k+1)}{2n}\right)=\frac{\pi}{2},\end{cases}$$

with respect to $\left(\varphi\left(\frac{\pi}{2n}\right),\ \varphi\left(\frac{3\pi}{2n}\right),\dots,\varphi\left(\frac{\pi(4n-1)}{2n}\right)\right)$

we obtain the value of the approximate solution $\varphi_n^*(t)$ of the equation (11)-(12) at the points $t_k=\frac{\pi(2k+1)}{2n}$, $k=\overline{0,2n-1}$.

In the following Table 1 we show the supremum-error

$$E=\sup_{k=\overline{0,2n-1}}\left|\varphi^*(t_k)-\varphi_n^*(t_k)\right|$$

and the mean-square-error

$$e=\left(\frac{1}{2n}\sum_{k=0}^{2n-1}\left|\varphi^*(t_k)-\varphi_n^*(t_k)\right|^2\right)^{\frac{1}{2}}$$

for different number of nodes $n=256$, $n=512$ and $n=1024$.

TABLE 1. THE SUPREMUM-ERROR AND MEAN-SQUARE-ERROR OF THE APPROXIMATE METHOD.

n	E	e
256	1.0149E-002	2.6891E-003
512	5.672E-003	1.3487E-003
1024	3.135E-003	6.755E-004

The following Figure 1 demonstrates the dependence of the supremum-error E on the number of nodes n and this relation shows the comparison of numerical solution and exact solution for the given hypersingular integral equation .

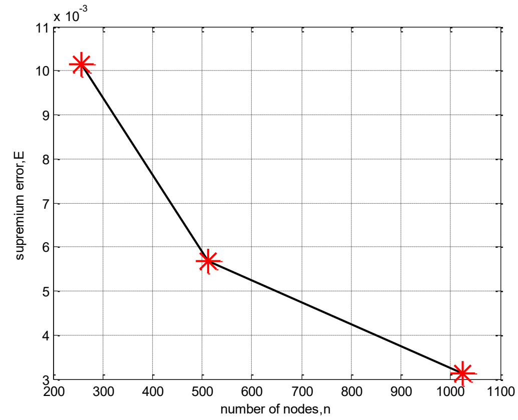

Fig. 1. Dependence of the supremum-error E on the number of nodes n.

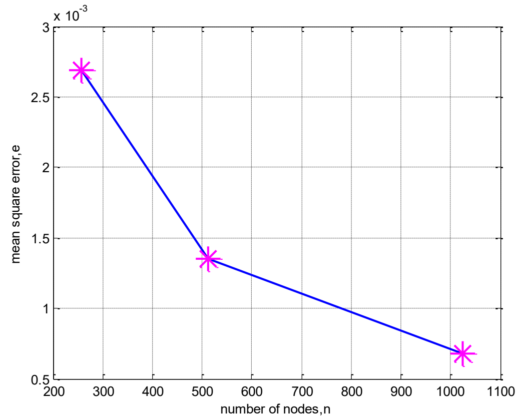

Fig. 2. Dependence of the mean-square error e on the number of nodes n.

The above-placed Figure 2 demonstrates the dependence of the mean-square error e on the number of nodes n. This relation shows the convergence rate of the numerical solution to the exact solution for the given hypersingular integral equation. Consequently, the numerical performance is consistent with the theoretical conclusions.

VI.CONCLUSION

The present paper has constructed and justified a new numerical method for the hypersingular integral equation. The advantages of this method are as follows: using this method, we obtain an exact estimate of the convergence rate for functions of the appropriate class, while other methods used earlier to approximate solutions of hypersingular integral equations require the study of the specific properties of the function classes under consideration; moreover, the estimate established in this paper yields more exact results in terms of the convergence rate than the methods used earlier; approximate solutions of the system of linear algebraic equations can be found explicitly (not at isolated points) and, therefore, given a fixed convergence rate, this method requires smaller computational resources.

REFERENCES

[1] Aliev, R.A., Gadjieva, Ch.A. "Approximate solution of hipersingulyar integral equations with Cauchy kernel", Transactions of NAS of Azerbaijan, Issue Mathematics, -2017, 37:1, -p.20-29.

[2] A.Yu.Anfinogenov, I.K.Lifanov, P.I.Lifanov, "On certain one and two dimensional hypersingular integral equations", Sbornik:Mathematics, 192:8 (2001), pp. 1089-1131.

[3] Gadjieva Ch.A.,"A new approximate method for solving hypersingular integral equations with Hilbert kernel", Proceedings of the IMM of NAS of Azerbaijan, 43:2, (2017), 316-329.

[4] Gadjieva Ch.A. "Approximation of hypersingular integral operators on Hölder spaces", Caspian Journal of Applied Mathematics, Ecology and Economics, **5:2** (2017), 96-105.

[5] Gadjieva Ch.A., "The Numerical Method for Approximation of Hilbert hypersingular integral and its justification with the numerical example", Proceedings of COIA2024, Turkey-2024,p.298-302

[6] Gadjieva, Ch. A.,"A New Approximate Method For Hypersingular Integral Operators With Hilbert Kernel", Journal of Contemporary Applied Mathematics,(2023), 13(1).

[7] Guseynov, O. M., & Mamedov, T. J. "Fundamental Solution of a Cauchy and Goursat Type Problem for Second Order Hyperbolic Equations and Related Optimal Control Problems", Journal of Contemporary Applied Mathematics, (2024), 14(1).

[8] I.V.Boykov, E.S.Ventsel, A.I.Boykova, An approximate solution of hypersingular integral equations, Applied Numerical Mathematics, 60:6 (2010), pp. 607-628.

[9] J. Hadamard, Lectures on Cauchy's problem in linear partial differential equations, Dover publication, New-York, (2003).

[10] Kaplan, A. G., "Applications of Laplace Transform Method to the Fractional Linear Integro Differential Equations", Journal of Contemporary Applied Mathematics, (2023), 13(1).

[11] Mamedov, I. G., & Abdullayeva, A. J. ,"Fundamental solution of one boundary value problem on the geometric middle of the domain for the 3d Bianchi integro-differential equation", Journal of Contemporary Applied Mathematics, (2024), 14(2), 26-37.

[12] Niyozov, I. E., Juraev, D. A., Efendiev, R. F., & Jalalov, M. J. "The Cauchy Problem for the System of Elasticity", Journal of Contemporary Applied Mathematics, (2024), 14(2).

[13] R.A.Aliev, A new constructive method for solving singular integral equations, Mathematical Notes, 79:(5-6) (2006), pp. 749-770.

[14] R.A.Aliev, A.F.Amrakhova, A constructive method for the solution of integral equations with Hilbert kernel, Proceedings of the Institute of Mathematics and Mechanics: Ural branch of the Russian Academy of Sciences, 18:4 (2012), pp. 14-25 (in Russian).

[15] R.A.Aliev, Ch.A.Gadjieva, Approximation of hypersingular integral operators with Cauchy kernel, Numerical Functional Analysis and Optimization, 37:9 (2016), pp. 1055-1065.

[16] R.A.Aliev, Ch.A.Gadjieva, Approximation of Hilbert tranforations.// Proceedings of the Institute of Mathematics and Mechanics of the Ural RAS, 2019, Vol. 25, №2, page 31-41

[17] Yuldashev, T. K., Mamedov, K. R., & Babayev, M. M. "Dependence on the parameters of the solution of a mixed problem for a nonlinear integro-differential equation of the fifth order with a degenerate kernel", Journal of Contemporary Applied Mathematics, (2024),14(2).