Numerical Solution of the Volterra Integral Analysis of the Measurement Function Using the Interpolation Method

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Abstract— Systems can be classified in many different ways. However, the most common classification of systems is whether the system is linear or not according to its structure. The transfer function method, which is a very simple approach, is generally used in the analysis of linear systems. In fact, almost all of the physical systems we see around us are nonlinear.

This article describes the essence of integral equations, the fields in which they are used, and the methods used in solving integral equations. The work on the selection and application of the interpolation method is outlined.

Keywords— Integral equations, solutions, interpolation methods, mathematical methods.

I. INTRODUCTİON

Integral equations are employed in numerous scientific and engineering disciplines, such as genetic engineering, scattering theory, potential theory, artificial intelligence, nanotechnology, thermodynamics, virology, and epidemiology, among others. When addressed through the integral equations method, these equations resemble the solutions of starting, boundary, and mixed value issues associated with Laplace equations, Helmholtz equations, electromagnetic wave equations, and other equations subject to Dirichlet conditions. Shukrallah et al. demonstrated methods for solving weakly singular Fredholm integral equations of the first class through diverse approaches and methodologies.

For nonlinear systems, they can be assumed to be linear by making some assumptions and omissions, and their analysis can be carried out with methods that can be applied to linear systems. However, as a result of this analysis, important behaviors specific to nonlinear systems such as jumps, bifurcations, and chaos cannot be observed and sufficient information about the system cannot be obtained. For this reason, many methods applied in the time and frequency dimensions have been developed for the analysis of nonlinear systems.

II.DESCRIPTION OF PROBLEM

The most suitable methods for the analysis of nonlinear systems are the methods applied in the frequency dimension. Because in the analysis methods in the time dimension, since the analysis is made for a certain time interval, nonlinear system behaviors cannot be observed. There are many methods that perform the analysis of nonlinear systems in the frequency dimension. The most common of these are the Volterra series method, the method of defining functions, and the generalized equilibrium equations method. All three of these methods are based on the Volterra model. The Volterra model was first proposed by Vito Volterra. Volterra showed that nonlinear systems can be defined with the infinite Volterra series he named. Accordingly, the output of a single-input system can be expressed with Volterra series. Brilliant showed that Volterra series can be applied not only to continuous-time systems but also to nonlinear memoryless systems, and methods were developed for the inversion, addition, multiplication, cascade combination of analytical systems, and the calculation of the results of simple feedback connections in an analytical structure, and it was proven that the resulting series converged to the desired point.

Ramdan et al. enhanced the homotopy perturbation approach by integrating it with a power series formula originally formulated by Taylor to derive accurate solutions to integral equations. Rahman et al. [8,9] addressed non-singular and singular Volterra equation of the second kind (SVK2) via the Galerkin weighted residual method utilizing the Chebyshev polynomial basis. Shahsavaran et al. [11] attained remarkable precision in the approximate solutions of the Cauchy singular integral problem using Gauss-Legendre quadrature points. This method resolved a singular integral equation by transforming it into a system of algebraic equations. Hou et al. [4] examined a fractional Jacobi-collocation spectral approach for solving SVK2. Chen et al. [20] devised a Jacobi-collocation spectral technique for addressing SVK2. Mori et al. [7] introduced a double exponential (DE) transformation to obtain a numerical solution for SVK2. Andras et al. [1] developed a novel approach grounded in a fixed-point theory, employing the iterated operator, and applied the fiber Picard operator theorem. Vainikko et al. [13] addressed SVK2 utilizing a Nystrom-type accuracy approach. Zhao et al. [12] employed smoothing transformations to regularize the integral equations, subsequently developing super implicit multistep collocation methods for solving SVK2. Utilizing Navot's quadrature and Simpson's rule, Araghi et al. [1,2] developed a numerical method to provide an approximate solution while addressing singularities. A range of numerical approaches exists for addressing nonlinear integral equations and integro-differential equations in various published sources [14], [15], This study will introduce an innovative interpolation method for addressing SVK2. The published articles [2,6], [5] cover interpolation techniques for resolving non-singular Volterra integral equations.

III. METHODOLOGY

Integral equations are equations in which the unknown function is used under the integral sign. The unknown function g(x), the kernel function K(x,t), a≤ x ≤b and a≤ t ≤b,

$$g(x) = f(x) + \int_a^b K(x,t)g(t)dt$$

When the functions f(x) and K(x,t) are known, the equation formed by the unknown function g(x) under the integral sign is called the integral equation.

Most basically, integral equations are classified as linear integral equations and nonlinear integral equations.

$$g(x) = \int_a^b K(x,t)g(t)dt \quad (1.1)$$

$$g(x) = f(x) + \int_a^b K(x,t)g(t)dt \quad (1.2)$$

$$g(x) = f(x) + \int_a^b K(x,t)[g(t)]^n dt \quad (1.3)$$

If only linear operations are performed on the function g(x) and the function g(x) is the unknown function, the integral equation is a linear integral equation. Equations (1.1) and (1.2) are examples of linear integral equations. The general condition for a linear operator can be written as $L[g(x)] = f(x)$ where L is the regular integral operator. For any constants c1 and c2,

$$L[c_1\, g_1(x) + c_2\, g_2(x)] = c_1\, L[\, g_1(x)] + c_2\, L[\, g_2(x)] \quad (1.4)$$

If the linear operator condition is satisfied, it is the most general form of the linear integral equation.

$$h(x)g(x) = f(x) + \lambda \int_a^b K(x,t)g(t)dt \quad (1.5)$$

The upper bound of the integral can be either changing or constant. The most general form of linear integral equations is (1.5) where f(x), h(x) are known functions, K(x,t) is the kernel function, λ is a nonzero parameter, and g(x) is the unknown function. Equation (1.3) is a nonlinear integral equation. Since it is of the nth degree, it does not satisfy the linear operator condition. A system consisting of many unknowns can be considered, where x and t are n-dimensional vectors and Ω is a region of n-dimensional space.

$$g(x) = f(x) + \int_\Omega K(x,t)g(t)dt \quad (1.6)$$

The case of singular and non-singular integral equations depends on the continuity of the kernel function K(x, t) on the intervals where the x and t variables of the kernel function K(x, t) are defined. If the kernel function K(x, t) is continuous on the intervals $a \le x \le b$ and $a \le x \le t$, then the integral equation is a non-singular integral equation on this interval. The integral equation (1.7) is an example of a non-singular integral equation.

$$g(x) = f(x) + \int_0^\pi \cos(xt)\, g(t)dt \quad (1.7)$$

If the kernel function K(x,t) is not continuous on the given interval and one or both of the boundaries are infinite, the integral equation is a singular integral equation. The following integral equations are examples of singular equations.

$$g(x) = f(x) + \int_0^x \ln\,(x-t)g(t)dt$$

$$g(x) = f(x) + \int_0^\infty \frac{\sin\, x}{x}\, g(t)dt$$

$$g(x) = f(x) + \int_{-\infty}^\infty \frac{g(t)}{e^{|x-t|}}dt$$

Fredholm and Volterra Integral Equations

Regardless of whether the integral equation is linear or homogeneous, the condition of the limits of the integral is decisive. If the lower limit of the integral is fixed and the upper limit is variable, the integral equation is called the Volterra integral equation. With known functions h(x), g(x) and f(x), the kernel functions K(x,t), the lower limit of the integral is the constant a and the upper limit is the variable x;

$$h(x) = \int_a^x K(x,t)g(t)dt$$

$$g(x) = \int_a^x K(x,t)g(t)dt$$

$$g(x) = f(x) + \int_a^x K(x,t)g(t)dt$$

$$h(x)g(x) = f(x) + \int_a^x K(x,t)g(t)dt$$

The Fredholm Integral Equation is the most general form of the Fredholm Integral Equation, where h(x), g(x), f(x) are known functions, K(x,t) is the kernel function and a, b are constants that form the lower and upper bounds of the integral.

$$h(x)g(x) = f(x) + \int_a^b K(x,t)g(t)dt$$

Fredholm integral equations are divided into types according to the case of the h(x) function.

i. Fredholm integral equation of the first kind, for h(x)=0,

$$f(x) + \int_a^b K(x,t)g(t)dt = 0$$

ii. The Fredholm integral equation of the second kind, for h(x)=1, becomes equation).

$$g(x) = f(x) + \int_a^b K(x,t)g(t)dt$$

In this article, we will examine the solution of Volterra integral equations using the interpolation method.

The interpolation method is used in solving integral equations due to the required accuracy and computational efficiency.

a. Linear Interpolation

Linear interpolation assumes that the functional x(t) is a piecewise linear function between consecutive points:

$$x(s) \approx x(t_i) + \frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}(s - t_i),\ t_i \le s \le t_{i+1}$$

b. Polynomial interpolation represents $x(t)$ as a polynomial of degree n passing through $n+1$ data points:

$$x(s) = \sum_{j=0}^{n} x(t_j)\ell_j(s)$$

where $\ell_j(s)$ are Lagrange basis polynomials:

$$\ell_j(s) = \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{s - t_k}{t_j - t_k}$$

c. Spline Interpolation

Spline interpolation fits piecewise polynomials (typically cubic) to the data points while ensuring smoothness at the boundaries:

$$x(s) = \begin{cases} P_1(s), & t_0 \le s \le t_1 \\ P_2(s), & t_1 \le s \le t_2 \\ \dots \end{cases}$$

The polynomials $P_i(s)$ satisfy continuity and smoothness conditions:

$$P_i(t_{i+1}) = P_{i+1}(t_{i+1}),\ P_i'(t_{i+1}) = P_{i+1}'(t_{i+1}),\ P_i''(t_{i+1}) = P_{i+1}''(t_{i+1}).$$

This assumes $x(s)$ remains constant within each interval:

$$x(s) \approx x(t_i),\ t_i \le s < t_{i+1}$$

Fig 1. Advantages of using interpolation methods with equations

Feature	Benefit
Simplification of Equations	Converts integral equations into algebraic equations for easier solving
Adaptable Accuracy	Adjustable interpolation degree for precision control
Computational Efficiency	Low computational cost for simple interpolation methods
Irregular Data Support	Handles unevenly spaced or non-standard datasets
Compatibility with Nonlinear Systems	Allows for iterative refinement in nonlinear scenarios
Scalability	Efficient for large-scale systems through segmentation or parallelization
Smooth and Realistic Solutions	Maintains continuity and realism with splines or low-degree polynomials

Source: Prepared by the author himself as a result of research

Fig 1. shows what the main advantages of the chosen method in solving the problem are.

The main advantages of using the interpolation method in solving Volterra integral equations are that it simplifies[10] the solution of equations due to the scale of the problem, increases computational efficiency, handles irregular data, and offers effective solutions for large-scale problems.

IV. Conclusion

In this article, we have examined the essence of Volterra integral equations and the methods and forms used in their solution. In this process, we have noted the importance of not only the interpolation method, but also other mathematical methods, and we have examined the mathematical solutions. We have also put forward solutions that have noted the main advantages associated with the use of the interpolation method.

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