On Convergence Rate to Invariant Measures in Continuous-Time Critical Markov Branching Systems with Possibly Infinite Variance and Allowing Immigration

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***Abstract*—Markov branching systems belong to an important class of stochastic processes widely used for modeling various phenomena in biology, physics, finance, and other fields. This system is characterized by a state that evolves over continuous time and the possibility of "branching" in each state, in which the system can transition to several different states, not just one. This property is especially important in studying population dynamics, disease spread, and models of random chains with multiple outcomes. Unlike simple Markov systems, branching systems require taking into account not only the transition probabilities between states but also the branching probabilities, which significantly complicates the mathematical description. We discuss the asymptotic properties of the transition functions of Markov branching systems allowing immigration.**

Keywords— branching systems, immigration, Markov chain, transition functions, generating functions, slowly varying function, invariant measures, convergence rate.

I. Introduction

Branching systems allowing immigration are an important generalization of standard branching systems that include external immigration events in addition to the natural branching dynamics. These systems are most suitable for modeling populations or systems where individuals are not only born within the population (via branching) but can also arrive from an external source (immigration), increasing the population size. These systems are called branching systems allowing immigration.

The key difference between a standard branching system and one with immigration is the presence of this external input, which can alter the long-term behavior of the system. In particular, for certain values of per-individual branching rate and immigration rate, the population might grow indefinitely (due to immigration), or it might tend toward a stable equilibrium size.

Thus, the population evolution is driven by two factors: the internal branching dynamics, which leads to the reproduction of individuals, and the external immigration, which contributes a constant influx of new individuals.

Branching systems with immigration are widely used in population biology, ecology, and epidemiology, where they model situations where a population not only grows through reproduction but also experiences external influences, such as migration, immigration of individuals, or the introduction of new species or pathogens; see [1]–[5], [10], [16].

This report deals with homogeneous continuous-time Markov branching systems allowing immigration (MBSI). In Section II, we describe the structure of MBSI mathematically. Section III contains the main results.

II. Structure of MBSI

Let $\mathbf{N}$ be the set of natural numbers and $\mathbf{N}_0 = \{0\} \bigcup \mathbf{N}$. Each individual existing at time $t \in \mathrm{T} := [0, \infty)$ in the population, independently of their history and of each other undergoes random transformations over an infinitesimal time interval $(t, t+\varepsilon)$ as $\varepsilon \downarrow 0$.

Specifically, as $\varepsilon \downarrow 0$,

- with probability $a_j \varepsilon + o(\varepsilon)$, the individual produces $j \in \mathbf{N}_0 \setminus \{1\}$ offspring;
- with probability $1 + a_1 \varepsilon + o(\varepsilon)$, the individual either survives without reproducing or produces exactly one descendant.

The branching rates $\left\{a_j, j \in \mathbf{N}_0\right\}$ satisfy $a_j \geq 0$ for $j \in \mathbf{N}_0 \setminus \{1\}$, with the predetermining condition

$$0 < a_0 < -a_1 = \sum_{k \in \mathbf{N}_0 \setminus \{1\}} a_k$$

that ensures the consistency of the system evolution.

In parallel, the immigration flow contributes to the population over the same time interval $(t, t+\varepsilon)$ as $\varepsilon \downarrow 0$, as follows:

- independently on the branching event that occurs in the system, $j \in \mathbf{N}$ new individuals arrive there with probability $b_j \varepsilon + o(\varepsilon)$;
- with probability $1 + b_0 \varepsilon + o(\varepsilon)$ no immigration occurs.

The immigration intensities $\left\{b_j, j \in \mathbf{N}_0\right\}$ satisfy $b_j \geq 0$ for $j \in \mathbf{N}$, while the condition $0 < -b_0 = \sum_{k \in \mathbf{N}} b_k < \infty$ ensures finite total immigration rates. Newly immigrated individuals immediately adopt the same reproduction dynamics as the resident population, with transformations governed by the intensities $\{a_j, j \in \mathbf{N}_0\}$. For a detailed description of this population growth mechanism; see [9], [11], [17].

So, MBSI is comprehensively characterized by two infinitesimal generating functions (GFs):

- $f(s) = \sum_{k\in\mathbf{N}_0} a_k s^k$ representing the reproductive dynamics, and
- $h(s) = \sum_{k\in\mathbf{N}_0} b_k s^k$ capturing immigration effects

for $s \in [0,1)$. Together, these functions uniquely determine the evolution of the population over continuous time.

Denoting $X(t)$ the population size at time $t \in \mathrm{T}$ in MBSI, we have an homogenous continuous-time Markov chain with state space $\mathrm{S} \subset \mathbf{N}_0$. Putting transition functions

$$p_{ij}(t) := \mathbf{P}_i\{X(t) = j\} = \mathbf{P}\{X(\tau + t) = j \mid X(\tau) = i\}$$

for any $\tau, t \in \mathrm{T}$, we recall that the corresponding GF $\mathrm{P}_i(t;s) := \sum_{j\in\mathrm{S}} p_{ij}(t) s^j$ has the form of

$$\mathrm{P}_i(t;s) = [F(t;s)]^i \exp\int_0^t h(F(u;s))\,du, \qquad (1)$$

where $F(t;s)$ is GF satisfies

- the backward $\dfrac{\partial F(t;s)}{\partial t} = f(F(t;s))$, and
- the forward $\dfrac{\partial F(t;s)}{\partial t} = f(s)\dfrac{\partial F(t;s)}{\partial s}$

Kolmogorov equations with the initial condition $F(0;s) = s$; for details on formula (1), see [6]–[9], [11]–[15].

It is known that $F(t;s) \to q$ as $t \to \infty$ uniformly in $s \in [0,d]$, $d < 1$, where q is the extinction probability of a single-individual hereditary dynasty; see [15]. Then the formula (1) implies that $\mathrm{P}_i(t;s)/\mathrm{P}(t;s) \to q$. Therefore, we can restrict our study to $\mathrm{P}(t;s) := \mathrm{P}_0(t;s)$ if $t \to \infty$. The population mean for $\mathbf{P}\{X(0) = 0\} = 1$ can then be calculated by differentiating equation (1), which gives:

$$\mathsf{E}X(t) := \left.\frac{\partial \mathrm{P}(t;s)}{\partial s}\right|_{s=1} = \begin{cases} \dfrac{h'(1^-)}{a}\left(e^{at} - 1\right) & if \quad a \neq 0 \\ h'(1^-)t & if \quad a = 0, \end{cases}$$

where the prime denotes the derivative, and $a = f'(1^-)$; from here on we use the notation $A(1^-) := \lim_{s\uparrow 1} A(s)$. The preceding formula indicates that the asymptotic behavior of $\mathsf{E}_i X(t)$ is governed by the sign of the parameter a, resulting in qualitatively different trajectories. In this context, the chain $\{X(t)\}$ is categorized as subcritical, critical, or supercritical when $a < 0$, $a = 0$ and $a > 0$, respectively.

III. Basic assumptions

Further, we will focus exclusively on the critical case and impose the following conditions on the GFs $f(s)$ and $h(s)$.

Let

$$f(s) = (1-s)^{1+\nu}\,\mathrm{L}\left(\frac{1}{1-s}\right) \qquad [f_\nu]$$

and

$$h(s) = -(1-s)^{\delta}\,\ell\left(\frac{1}{1-s}\right) \qquad [h_\delta]$$

for all $s \in [0,1)$, where $\nu, \delta \in (0,1)$ and, the functions $\mathrm{L}(\cdot)$ and $\ell(\cdot)$ vary slowly at infinity in the sense of Karamata's concept of regular variation. The basic assumptions state that the offspring distribution belongs to the domain of attraction of the $(1+\nu)$ -stable law, whereas the immigration distribution belongs to the domain of attraction of the δ -stable law. By the criticality of the system, the assumption $[f_\nu]$ implies that $2b := f''(1^-) = \infty$. If $b < \infty$ then $[f_\nu]$ holds for $\nu = 1$ and $\mathrm{L}(t) \to b$ as $t \to \infty$. Similarly, the GF $h(s)$ of the form $[h_\delta]$ generates the immigration law, which has the δ -order moment. However, if $h'(1^-) < \infty$, then $[h_\delta]$ is satisfied with $\delta = 1$ and $\ell(t) \to h'(1^-)$ as $t \to \infty$.

It was established in [7] that S is positive-recurrent if $\gamma := \delta - \nu > 0$, and transient if $\gamma < 0$. In the special case where $\gamma = 0$, it follows that $h(s) = f'(s)$ and $\mathrm{L}(t) \to 1+\nu$ as $t \to \infty$. Under this condition, a different population process, known as the Markov Q-process, replaces the MBSI. For details on the Markov Q-process, we refer the reader to [1], [4], [6], [8] and [14].

Next, it is known that every slowly varying function at infinity $L(\cdot)$ is defined by the condition $L(\lambda t) \sim L(t)$ for each $\lambda > 0$, meaning that the ratio $L(\lambda t)/L(t)$ approaches 1 at a rate characterized by some infinitesimal function as $t \to \infty$, uniformly in λ. Thus we can write

$$\frac{\mathrm{L}(\lambda t)}{\mathrm{L}(t)} = 1 + \omega_\lambda(t) \qquad [\mathrm{L}_\nu]$$

for each $\lambda > 0$, where $\omega_\lambda(t)$ is a positive decreasing function such that $\omega_\lambda(t) \to 0$ as $t \to \infty$ uniformly in λ. If $\omega_\lambda(t)$ is known, then the function $\mathrm{L}(\cdot)$ is said to belong to the class of slowly varying functions with remainder $\omega_\lambda(t)$ (see [2, p. 185]), which we denote as $\mathrm{L}(\cdot) \in \mathrm{SV}_\infty(\omega)$. Whenever we use the condition $[\mathrm{L}_\nu]$, we will assume that

$$\omega_\lambda(t) = \mathrm{O}\left(\frac{\mathrm{L}(t)}{t^\nu}\right) \quad as \quad t \to \infty.$$

Similarly, we also impose the condition

$$\frac{\ell(\lambda t)}{\ell(t)} = 1 + \mathrm{O}(\beta(t)) \quad as \quad t \to \infty \qquad [\ell_\delta]$$

for each $\lambda > 0$, where

$$\beta(t) = \mathrm{O}\left(\frac{\ell(t)}{t^{\delta}}\right) \quad as \quad t \to \infty .$$

The above conditions $[f_\nu]$, $[h_\delta]$, $[\mathrm{L}_\nu]$ and $[\ell_\delta]$ constitute our ***Basic assumptions***.

The paper [7] demonstrated that the asymptotic behavior of the transition functions $p_{ij}(t)$ is determined by the sign of the parameter $\gamma := \delta - \nu$. It was shown that if $\gamma < 0$, there exists a certain function $T(t)$ such that the limit function

$$\pi(s) := \lim_{t\to\infty} e^{T(t)} \mathrm{P}(t;s)$$

has an explicit form depending on GFs $f(s)$ and $h(s)$. The power series representation $\pi(s) = \sum_{k\in\mathrm{S}} \pi_k s^k$ generates an invariant measure $\{\pi_j\}$ for the system $\{X(t)\}$.

In this report, we focus on determining the rate of convergence of $e^{T(t)}\mathrm{P}(t;s)$ to $\pi(s)$.

IV. Main Result

Let $\gamma < 0$ and define the function:

$$\mathsf{L}(t) := \frac{\ell(t)}{\mathrm{L}(t)} \qquad \text{and} \qquad \tau(t) := \frac{(\nu t)^{1/\nu}}{\mathrm{N}(t)},$$

where $\mathrm{N}(t)$ is slowly varying function such that

$$\mathrm{N}^{\nu}(t)\mathrm{L}\left(\tau(t)\right) \to 1 \quad as \quad t \to \infty .$$

Lemma A [6]. *Under Basic assumption* $\mathsf{L}(\cdot) \in \mathrm{SV}_\infty(\omega)$ *and there exist* $C_{\mathsf{L}} := \mathsf{L}(\infty) = C_\ell / C_{\mathrm{L}}$ *, where* $C_\ell = \ell(\infty)$ *,* $C_{\mathrm{L}} = \mathrm{L}(\infty)$*. Moreover*

$$\mathsf{L}(t) = C_{\mathsf{L}} + \left(1/t^{\delta}\right) \quad as \quad t \to \infty . \qquad (2)$$

The results proved in [7] implies that if $\gamma < 0$ then

$$-\left(\tau(t)\right)^{|\gamma|} \ln p_{00}(t) = \frac{1}{|\gamma|}\mathsf{L}_\gamma\left(\tau(t)\right), \qquad (3)$$

where $\mathsf{L}_\gamma(t)/\mathsf{L}(t) \to 1$ as $t \to \infty$. Asymptotic formula (3) suggests that, for $\gamma < 0$, the primary focus should be directed toward investigating the asymptotic behavior as $t \to \infty$ of the function $e^{T(t)}\mathrm{P}(t;s)$, with $T(t) := \left(\tau(t)\right)^{|\gamma|}$, rather than considering the function $\mathrm{P}(t;s)$.

Our principal result is in the following theorem.

Theorem 1. *Let Basic assumptions hold and* $\gamma < 0$. *If* $\mu := 2\delta - \nu > 0$ *and* $C_{\mathsf{L}} = |\gamma|$ *in (2), then*

$$\zeta(t;s) := 1 - \frac{1}{\pi(s)} e^{T(t)}\mathrm{P}(t;s) \to 0 \qquad as \quad t \to \infty ,$$

where the limiting-invariant GF $\pi(s) := \lim_{t\to\infty} e^{T(t)}\mathrm{P}(t;s)$ *has the form*

$$\pi(s) = \exp\left\{\frac{1}{(1-s)^{|\gamma|}} + \int_s^1 \left[\frac{h(x)}{f(x)} + \frac{|\gamma|}{(1-x)^{1+|\gamma|}}\right]dx\right\}$$

Moreover

$$\zeta(t;s) = \frac{1}{\mu}\frac{C_\ell}{\left(C_{\mathrm{L}}\right)^{\mu/\nu}}\frac{1}{(\nu t)^{\mu/\nu}}\left(1 + \sigma(t;s)\right),$$

where the constants C_ℓ *and* C_{L} *are specified in Lemma A and* $\sigma(t;s) \to 0$ *as* $t \to \infty$ *uniformly in* $s \in [0,d]$, $d < 1$.

Remark 1. Further discussion reveals that the function $\pi(s) = \sum_{k\in\mathrm{S}} \pi_k s^k$ satisfies the Schröder functional equation

$$\pi(s) = \mathrm{P}(\tau;s)\pi\left(F(\tau;s)\right) \text{ for any } \tau \in \mathrm{T} .$$

Remark 2. The above theorem refines the corresponding result in [7] by explicitly characterizing the decay behavior of the remainder term $\zeta(t;s)$.

Remark 3. By applying the continuity theorem for power series in Theorem 1, one can obtain the asymptotic expansion with the remainder term of transition functions $p_{ij}(t)$ showing their convergence to invariant measures $\{\pi_j\}$.

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