On an Inverse Problem of Determining the Time Derivative Coefficient in a Parabolic Equation

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Abstract **–This paper studies a nonlocal inverse boundary-value problem for a second-order parabolic equation. The primary objective is to identify the time derivative coefficient in the parabolic equation using additional measurements. To analyze the classical solvability of the original problem, the considered problem is reduced to an auxiliary inverse boundary-value problem. Furthermore, the existence and uniqueness theorem for the solution to the equivalent problem is proved. Finally, using this equivalence, sufficient conditions for the existence and uniqueness of the classical solution to the inverse problem for the second-order parabolic equation are established.**

Keywords – inverse problem, parabolic equation, classical solution, existence, uniqueness.

I. Introduction and Problem Statement

The study of inverse boundary-value problems for parabolic equations is one of the classical problems of the theory of partial differential equations and has always attracted the attention of mathematicians.

The problem of determining the right-hand side of a differential equation or recovering unknown coefficients from given data is called an inverse boundary-value problem in the theory of mathematical physics. These problems are widely encountered in fields such as mechanics, physics, geology, engineering, medical imaging, and etc.

Currently, numerous studies are devoted to the investigation of the solvability of various inverse problems for parabolic equations (see, for example, [1], [3]-[11] and the references therein).

It is worth noting that one classe of qualitatively new problems are problems with nonlocal conditions. The term “nonlocal conditions” was first introduced by A.A. Dezin [2]. Nonlocal boundary-value problems are usually called problems in which, by together specifying the values of the solution or its derivatives on a fixed part of the boundary, a relationship is established between these values and the values of the same functions on other internal or boundary manifolds. Nonlocal conditions may arise in a situation where the boundary of the domain of a real process is not available for direct measurements, but it is possible to obtain some additional information about the phenomenon under study at the internal points of the domain. Often such information comes in the form of some average values of the desired solution.

In the present paper, we address an inverse problem for identifying the time derivative coefficient in a parabolic equation using some additional measurements. It should be noted that the problem formulation and the investigation techniques used in this study differ from those in previous works.

Let $T > 0$ be a fixed time moment and let D_T be a rectangular domain in the xt - plane bounded by $0 < x < 1,\ 0 < t < T$. Consider the problem of determining the unknown functions $u(x,t)$ and $p(t) > 0$ such that the pair $\{u(x,t), p(t)\}$ satisfies the following one-dimensional parabolic equation

$$p(t)u_t(x,t) = u_{xx}(x,t) + f(x,t)\ \ (x,t) \in D_T, \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x)\ \ (0 \le x \le 1), \tag{2}$$

the periodic boundary condition

$$u(0,t) = u(1,t)\ \ (0 \le t \le T), \tag{3}$$

the integral condition

$$\int_0^1 u(x,t)dx = 0\ \ (0 \le t \le T), \tag{4}$$

and additional condition

$$u\left(\frac{1}{2}, t\right) = q(t)\ \ (0 \le t \le T), \tag{5}$$

where $f(x,t)$, $\varphi(x)$, and $q(t)$ are known functions.

Definition. The pair $\{u(x,t), p(t)\}$ is said to be a classical solution of problem (1) - (5), if the functions $u(x,t)$ and $p(t)$ satisfy the following conditions:

i) The function $u(x,t)$ and its derivatives $u_t(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, are continuous in the domain $\overline{D}_T$.

ii) The function $p(t)$ is positive and continuous on the interval $[0,T]$.

iii) Equation (1) and conditions (2) - (5) are satisfied in the classical (usual) sense.

Theorem 1. Assume that $f(x,t)\in C(\overline{D}_T)$, $\varphi(x)\in C[0,1]$, $\int_0^1 f(x,t)dx=0$ $(0\le t\le T)$, $q(t)\in C^1[0,T]$, and the compatibility conditions

$$\int_0^1 \varphi(x)dx=0$$

and

$$\varphi\left(\frac{1}{2}\right)=q(0)$$

hold. Then, the problem of finding a classical solution of (1)-(5) is equivalent to the problem of determining the functions $u(x,t)\in C^{2,1}(\overline{D}_T)$ and $0<p(t)\in C[0,T]$ satisfying equation (1), conditions (2) and (3), and the conditions

$$u_x(0,t)=u_x(1,t)\ (0\le t\le T), \tag{6}$$

$$p(t)q'(t)=u_{xx}\left(\frac{1}{2},t\right)+f\left(\frac{1}{2},t\right)\ (0\le t\le T). \tag{7}$$

II. Existence and Uniqueness of a Solution to the Inverse Problem

It is known that the system

$$1,\cos\lambda_1 x,\ \sin\lambda_1 x,\ldots,\cos\lambda_k x,\ \sin\lambda_k x,\ldots, \tag{8}$$

forms a basis in $L_2(0,1)$, where $\lambda_k=2k\pi\ (k=0,1,\ldots)$.

Since the system (8) forms a basis in $L_2(0,1)$, we will seek the first component of the classical solution $\{u(x,t),p(t)\}$ to the problem (1)-(3), (6), (7) in the following form:

$$u(x,t)=\sum_{k=0}^{\infty}u_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_k x\ \ (\lambda=2k\pi), \tag{9}$$

where

$$u_{10}(t)=\int_0^1 u(x,t)dx,$$

$$u_{1k}(t)=2\int_0^1 u(x,t)\cos\lambda_k x dx\ \ (k=1,2,\ldots),$$

$$u_{2k}(t)=2\int_0^1 u(x,t)\sin\lambda_k x dx\ \ (k=1,2,\ldots).$$

Applying the formal Fourier method to determine the unknown coefficients $u_{1k}(t)\ (k=0,1,\ldots)$ and $u_{2k}(t)\ (k=1,2,\ldots)$ of the function $u(x,t)$, from (1) and (2), we obtain

$$p(t)u_{10}'(t)=f_{10}(t)\ \ (0\le t\le T), \tag{10}$$

$$p(t)u_{ik}'(t)+\lambda_{ik}^2 u_{ik}(t)=f_{ik}(t)\quad (i=1,2;\ k=1,2\ldots;\ 0\le t\le T), \tag{11}$$

$$u_{1k}(0)=\varphi_{1k}\ \ (k=0,1,2\ldots), \tag{12}$$

$$u_{2k}(0)=\varphi_{2k}\ \ (k=1,2\ldots), \tag{13}$$

where

$$f_{10}(t)=\int_0^1 f(x,t)dx,\ f_{1k}(t)=2\int_0^1 f(x,t)\cos\lambda_k x dx,$$

$$f_{2k}(t)=2\int_0^1 f(x,t)\sin\lambda_k x dx,\ \ k=1,2,\ldots,$$

$$\varphi_{10}=\int_0^1\varphi(x)dx,\ \varphi_{1k}(t)=2\int_0^1\varphi(x)\cos\lambda_k x dx,$$

$$\varphi_{2k}(t)=2\int_0^1\varphi(x)\sin\lambda_k x dx,\ \ k=1,2,\ldots.$$

Solving problem (10) - (13) yields

$$u_{10}(t)=\varphi_{10}+\int_0^t\frac{1}{p(\tau)}f_{10}(\tau)d\tau, \tag{14}$$

$$u_{ik}(t)=e^{-\int_0^t\frac{\lambda_k^2}{p(s)}ds}\varphi_{ik}+$$

$$+\int_0^t\frac{1}{p(\tau)}f_{ik}(\tau)e^{-\int_\tau^t\frac{\lambda_k^2}{p(s)}ds}d\tau\ \ (i=1,2;\ k=1,2,\ldots). \tag{15}$$

After substituting the expressions for $u_{1k}(t)\ (k=0,1,\ldots)$ and $u_{2k}(t)\ (k=1,2,\ldots)$ into equation (9), we find

$$u(x,t)=\varphi_{10}+\int_0^t\frac{1}{p(\tau)}f_{10}(\tau)d\tau+$$

$$+\sum_{k=1}^{\infty}\left(e^{-\int_0^t\frac{\lambda_k^2}{p(s)}ds}\varphi_{1k}+\int_0^t\frac{1}{p(\tau)}f_{1k}(\tau)e^{-\int_\tau^t\frac{\lambda_k^2}{p(s)}ds}d\tau\right)\cos\lambda_k x+$$

$$+\sum_{k=1}^{\infty}\left(e^{-\int_0^t\frac{\lambda_k^2}{p(s)}ds}\varphi_{2k}+\int_0^t\frac{1}{p(\tau)}f_{2k}(\tau)e^{-\int_\tau^t\frac{\lambda_k^2}{p(s)}ds}d\tau\right)\sin\lambda_k x. \tag{16}$$

Now, using (9) from (7) we get

$$p(t)=[q'(t)]^{-1}\left\{\sum_{k=1}^{\infty}(-1)^{k+1}\lambda_k^2 u_{1k}(t)+f\left(\frac{1}{2},t\right)\right\}. \tag{17}$$

Taking into consideration (15), from (17) we obtain

$$p(t)=[q'(t)]^{-1}\left\{f\left(\frac{1}{2},t\right)+\sum_{k=1}^{\infty}(-1)^{k+1}\lambda_k^2\times\right.$$

$$\left.\times\left(e^{-\int_0^t\frac{\lambda_k^2}{p(s)}ds}\varphi_{1k}+\int_0^t\frac{1}{p(\tau)}f_{1k}(\tau)e^{-\int_\tau^t\frac{\lambda_k^2}{p(s)}ds}d\tau\right)\right\}. \tag{18}$$

Using the formula of integration by parts, we have

$$\int_0^t\frac{f_{1k}(\tau)}{p(\tau)}e^{-\int_\tau^t\frac{\lambda_k^2}{p(s)}ds}d\tau=\frac{f_{1k}(t)}{\lambda_k^2}-$$

$$-\frac{f_{1k}(0)}{\lambda_k^2}e^{-\int_\tau^t \frac{\lambda_k^2}{p(s)}ds}-\frac{1}{\lambda_k^2}\int_0^t f'_{1k}(\tau)e^{-\int_\tau^t \frac{\lambda_k^2}{p(s)}ds}d\tau. \qquad (19)$$

By virtue of relation (19), we obtain from (18) that

$$p(t)=\frac{f\left(\frac{1}{2},t\right)}{q'(t)}+\frac{1}{q'(t)}\sum_{k=1}^{\infty}(-1)^{k+1}\lambda_k^2\left\{\frac{f_{1k}(t)}{\lambda_k^2}+\right.$$

$$\left.+\left(\varphi_{1k}-\frac{f_{1k}(0)}{\lambda_k^2}\right)e^{-\int_0^t \frac{\lambda_k^2}{p(s)}ds}-\frac{1}{\lambda_k^2}\int_0^t f'_{1k}(\tau)e^{-\int_\tau^t \frac{\lambda_k^2}{p(s)}ds}d\tau\right\}. \qquad (20)$$

Thus, the solution to the problem (1)-(3), (6), (7) has been reduced to solving the system (16), (20) for the unknown functions $u(x,t)$ and $p(t)$.

Theorem 2. Assume that the data of problems (1)-(3), (6) and (7) satisfy the following conditions:

A) $q(t)\in C^1[0,T],\ q'(t)\neq 0,\ 0\le t\le T$;

B) $f\left(\frac{1}{2},t\right)q'(t)>0,\ q'(t)(-1)^{k+1}f_{1k}(t)>0,$

$$q'(t)(-1)^{k+1}\left(\varphi_{1k}-\frac{f_{1k}(0)}{\lambda_k^2}\right)>0,$$

$$q'(t)(-1)^{k+1}f'_{1k}(t)<0,\ 0\le t\le T;\ k=1,2,\dots;$$

C) $\varphi''(x)\in C[0,1],\ \varphi'''(x)\in L_2(0,1),$

$$\varphi(0)=\varphi(1),\ \varphi'(0)=\varphi'(1),\ \varphi''(0)=\varphi''(1);$$

D) $f(x,t), f_x(x,t), f_{xx}(x,t), f_t(x,t), f_{tx}(x,t), f_{txx}(x,t)\in C(D_T),$

$$f_{xxx}(x,t),\ f_{txxx}(x,t)\in L_2(D_T),$$

$$f(0,t)=f(1,t),\ f_x(0,t)=f_x(1,t),\ f_{xx}(0,t)=f_{xx}(1,t).$$

Then the problem (1)-(3), (6), (7), has a unique solution for sufficiently small values of T.

Hence, from Theorem 1 and Theorem 2, it follows the validity of the following assertion.

Theorem 3. Suppose that all assumptions of Theorem 2 and the compatibility conditions

$$\int_0^1\varphi(x)dx=0,\ \varphi\left(\frac{1}{2}\right)=q(0),$$

holds. Then the problem (1) - (5) has a unique classical solution for sufficiently small values of T.

CONCLUSION

In this paper, a nonlocal inverse boundary-value problem for a second-order parabolic equation is considered, and the time derivative coefficient is determined using certain overdetermination conditions. Moreover, through some trivial transformations, the original problem is reduced to an auxiliary equivalent problem, and the existence and uniqueness theorem for the solution of the equivalent problem is established. Furthermore, using this equivalence, the existence and uniqueness of the classical solution to the inverse problem for the second-order parabolic equation are proved. These findings contribute to a deeper understanding of nonlocal inverse boundary-value problems and provide a solid foundation for future research in this area.

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