Cauchy Problem for a Poroelasticity Model Described by Inhomogeneous Equations with Three Elasticity Parameters

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***Abstract*— In this paper the Cauchy problem for a one-dimensional inhomogeneous system of poroelasticity equations in the reversible hydrodynamic approximation was considered. The solution of the Cauchy problem was found by the method of characteristics, and this solution is represented by the form d'Alembert formula. The problem models the behavior of a porous elastic medium, where both solid and fluid phases interact. This approach allows for the calculation of wave propagation in the medium.**

Keywords*— *mathematical model, Cauchy problem, d'Alembert formula, slow wave, poroelasticity

I. Introduction

In applied wave process theory, key factors such as porosity, fluid saturation, and hydrodynamic background play a crucial role. These factors are important in exploration geophysics, in oil well development and optimization of wave action parameters in oil and gas fields. The goal is to improve production efficiency. Moreover, these considerations are an important part of monitoring efforts aimed at understanding focal zone properties, which is important for earthquake prediction.

A mathematical model was constracduced in [9], to describe the nonlinear wave propagation in a porous, elastically deformable medium saturated with liquid. This model was introducted by three foundational principles: conservation laws, the Galilean invariance principle, and the consistency of the motion equations of the saturating liquid with thermodynamic equilibrium conditions. These principles ensure a comprehensive framework that captures the complex interactions between the porous medium and the saturating fluid, providing insights into the dynamic behavior of such systems:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\mathbf{j} = 0, \mathbf{j} = \rho_s \mathbf{u}_1 + \rho_l \mathbf{u}_2$$

$$\frac{\partial S}{\partial t} + \mathrm{div}\left(\frac{S}{\rho}\mathbf{j}\right) = 0, \rho = \rho_l + \rho_s,$$

$$\frac{\partial \rho_l}{\partial t} + \mathrm{div}(\rho_l \mathbf{u}_2) = 0,$$

$$\frac{\partial g_{ik}}{\partial t} + g_{kj}\partial_i u_{1j} + g_{ij}\partial_k u_{1j} + u_{1j}\partial_j g_{ik} = 0,$$

$$\frac{\partial e}{\partial t} + \mathrm{div}\mathbf{Q} = 0, \qquad \rho_s = const\sqrt{\det(g_{ik})},$$

$$\frac{\partial j_i}{\partial t} + \partial_k\left(\rho_s u_{1i}u_{1k} + \rho_s u_{2i}u_{2k} + p\delta_{ik} + h_{ij}g_{jk}\right) = 0,$$

$$Q_k = \left(\hat{\mu} + \frac{u_2^2}{2} + \frac{TS}{\rho}\right) j_k + \rho_s(\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2)u_{1k} + u_{1i}h_{km}g_{mi},$$

$$\frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_2, \nabla)\mathbf{u}_2 = -\frac{\nabla p}{\rho} + \frac{\rho_s}{2\rho}\nabla(\mathbf{u}_1 - \mathbf{u}_2)^2 - \frac{h_{ik}}{2\rho}\nabla g_{ik}.$$

In this system,
$\mathbf{u}_1$ represents the velocity of the elastic porous medium,
$\mathbf{u}_2$ is the velocity of the saturating liquid,
the total density of the continuum is $\rho = \rho_l + \rho_s$
where
ρ_s is the partial density of the porous medium,
ρ_l is the partial density of the liquid,
g_{ik} is the elastic strain tensor,
h_{ik} is the stress tensor,
e and S the energy and entropy of unit volume are respectively,
$\hat{\mu}$ is the chemical potential,
T is the temperature and p is the pressure and
j_0 is the relative momentum.

Under these conditions, the system adheres to the first law of thermodynamics

$$de_0 = TdS + \hat{\mu}d\rho + (\mathbf{u}_1 - \mathbf{u}_2, dj_0) + \frac{1}{2}h_{ik}dg_{ik}.$$

In the work [12] a solution to the Cauchy problem for a homogeneous poroelasticity system was obtained. In this paper, a formula for solution to the Cauchy problem for a one-dimensional non-homogeneous system of poroelasticity equations described by three elasticity parameters in a reversible hydrodynamic approximation is obtained.

II. Statement of the Problem

We consider in the following the linear system of process of wave propagation in a porous medium in a reversible approximation. The system is described by a one-dimensional non-homogeneous system of equations taking into account mass forces $\boldsymbol{F}$ [9, 13, 14].

$$\frac{\partial^2 u_1(x,t)}{\partial t^2} - a_{11}\frac{\partial^2 u_1(x,t)}{\partial x^2} - a_{12}\frac{\partial^2 u_2(x,t)}{\partial x^2} = F_1(x,t), \qquad (1)$$

$$\frac{\partial^2 u_2(x,t)}{\partial t^2} - a_{21}\frac{\partial^2 u_1(x,t)}{\partial x^2} - a_{22}\frac{\partial^2 u_2(x,t)}{\partial x^2} = F_2(x,t), \qquad (2)$$

where $\rho_s = \rho_s^f(1 - d_0)$ is the partial density of elastic porous body, $\rho_l = \rho_l^f d_0$ is the partial density of liquid, d_0– is the porosity, ρ_s^f and ρ_l^f are the physical densities porous body and liquid respectively,

$$a_{11}=\frac{\lambda+2\mu}{\rho_s}+\left(\rho\alpha_3+\frac{\lambda+2\mu/3}{\rho^2}\right)\rho_s-\frac{\lambda+2\mu/3}{\rho},$$

$$a_{12}=\left(\rho^2\alpha_3+\frac{\lambda+2\mu/3}{\rho^2}-\frac{\lambda+2\mu/3}{\rho_s}\right)\frac{\rho_l}{\rho},$$

$$a_{21}=\left(\rho^2\alpha_3+\frac{\lambda+2\mu/3}{\rho}\right)\frac{\rho_l}{\rho}-\frac{\lambda+2\mu/3}{\rho},$$

$$a_{22}=\left(\rho^2\alpha_3+\frac{\lambda+2\mu/3}{\rho}\right)\frac{\rho_l}{\rho},$$

α_3, λ, μ – are the elastic parameters of the porous medium.

Let us consider the Cauchy problem for the system (1), (2) with the following Cauchy data [10]:

$$u_1|_{t=0}=\varphi_1(x), \qquad \left.\frac{\partial u_1}{\partial t}\right|_{t=0}=\psi_1(x) \tag{3}$$

$$u_2|_{t=0}=\varphi_2(x), \qquad \left.\frac{\partial u_2}{\partial t}\right|_{t=0}=\psi_2(x) \tag{4}$$

III. MAIN PART

It is practical to redefine the functions u_1, F_1, u_2, and F_2 in terms of new variables $\tilde{u}_1$, $\tilde{F}_1$, $\tilde{u}_2$, and $\tilde{F}_2$. These transformations are expressed as:

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}=T\begin{pmatrix}\tilde{u}_1\\\tilde{u}_2\end{pmatrix}, \quad \begin{pmatrix}F_1\\F_2\end{pmatrix}=T\begin{pmatrix}\tilde{F}_1\\\tilde{F}_2\end{pmatrix}$$

where

$$T=\begin{pmatrix}c_{l_1}^2-a_{22} & c_{l_2}^2-a_{22}\\ a_{21} & a_{21}\end{pmatrix}$$

$$c_{l_1}^2=B_*\left(1+\sqrt{1-\frac{b_*}{B_*^2}}\right),\ c_{l_2}^2=B_*\left(1-\sqrt{1-\frac{b_*}{B_*^2}}\right),$$

$$B_*=\frac{\alpha_3\rho^2}{2}+\frac{\lambda+2\mu/3}{2\rho}\left(\frac{\lambda+2\mu}{\lambda+\frac{2}{3}\mu}\frac{\rho}{\rho_s}-1\right),$$

$$b_*=(\lambda+2\mu)\left(\frac{1}{\rho_s}-\frac{1}{\rho}\right)\left[\frac{\lambda+2\mu/3}{\rho}\left(1-\frac{\lambda+\frac{2}{3}\mu}{\lambda+2\mu}\right)+\alpha_3\rho^2\right].$$

Then, the system of equations (1) and (2) is equivalent to the two-string equations

$$\frac{\partial^2\tilde{u}_1}{\partial t^2}-c_{l_1}^2\frac{\partial^2\tilde{u}_1}{\partial x^2}=\tilde{F}_1, \tag{5}$$

$$\frac{\partial^2\tilde{u}_2}{\partial t^2}-c_{l_2}^2\frac{\partial^2\tilde{u}_2}{\partial x^2}=\tilde{F}_2. \tag{6}$$

Let us consider inhomogeneous systems (5), (6) with zero Cauchy data:

$$\tilde{u}_1|_{t=0}=0, \qquad \left.\frac{\partial\tilde{u}_1}{\partial t}\right|_{t=0}=0, \tag{7}$$

$$\tilde{u}_2|_{t=0}=0, \qquad \left.\frac{\partial\tilde{u}_2}{\partial t}\right|_{t=0}=0. \tag{8}$$

Functions

$$v_1(t,x,\tau)=\frac{1}{c_{l_1}}\int_{x-c_{l_1}t+c_{l_1}\tau}^{x+c_{l_1}t-c_{l_1}\tau}\tilde{F}_1(\tau,\xi)\,d\xi,$$

$$v_2(t,x,\tau)=\frac{1}{c_{l_2}}\int_{x-c_{l_2}t+c_{l_2}\tau}^{x+c_{l_2}t-c_{l_2}\tau}\tilde{F}_2(\tau,\xi)\,d\xi,$$

are solutions of a homogeneous system

$$\frac{\partial^2 v_1}{\partial t^2}-c_{l_1}^2\frac{\partial^2 v_1}{\partial x^2}=0, \tag{9}$$

$$\frac{\partial^2 v_2}{\partial t^2}-c_{l_2}^2\frac{\partial^2 v_2}{\partial x^2}=0, \tag{10}$$

with Cauchy initial data

$$v_1|_{t=\tau}=0, \qquad \left.\frac{\partial v_1}{\partial t}\right|_{t=\tau}=\tilde{F}_1(\tau,x), \tag{11}$$

$$v_2|_{t=\tau}=0, \qquad \left.\frac{\partial v_2}{\partial t}\right|_{t=\tau}=\tilde{F}_2(\tau,x). \tag{12}$$

The functions

$$\tilde{u}_1(t,x)=\int_0^t v_1(t,x,\tau)\,d\tau,\ \ \tilde{u}_2(t,x)=\int_0^t v_2(t,x,\tau)\,d\tau \tag{13}$$

are solutions of problem (5) - (8) according to Duhamel's principle.

The solution to the Cauchy problem for a non-homogeneous system of poroelasticity equation in the entire space in the one-dimensional case generalizes the formulas obtained from [12] for a homogeneous system and is presented in the form:

$$u_1(t,x)=\frac{c_{l_1}^2-a_{22}}{2\left(c_{l_1}^2-c_{l_2}^2\right)}\left(\varphi_1(x+c_{l_1}t)+\varphi_1(x-c_{l_1}t)+\frac{1}{c_{l_1}}\int_{x-c_{l_1}t}^{x+c_{l_1}t}\psi_1(\xi)d\xi\right)-$$
$$\frac{(c_{l_1}^2-a_{22})(c_{l_2}^2-a_{22})}{2a_{21}\left(c_{l_1}^2-c_{l_2}^2\right)}\left(\varphi_2(x+c_{l_1}t)+\varphi_2(x-c_{l_1}t)-\frac{1}{c_{l_1}}\int_{x-c_{l_1}t}^{x+c_{l_1}t}\psi_2(\xi)d\xi\right)+$$
$$\frac{(c_{l_1}^2-a_{22})(c_{l_2}^2-a_{22})}{2a_{21}\left(c_{l_1}^2-c_{l_2}^2\right)}\left(\varphi_2(x+c_{l_2}t)+\varphi_2(x-c_{l_2}t)+\frac{1}{c_{l_2}}\int_{x-c_{l_2}t}^{x+c_{l_2}t}\psi_2(\xi)d\xi\right)-$$
$$\frac{c_{l_2}^2-a_{22}}{2\left(c_{l_1}^2-c_{l_2}^2\right)}\left(\varphi_1(x+c_{l_2}t)+\varphi_1(x-c_{l_2}t)+\frac{1}{c_{l_2}}\int_{x-c_{l_2}t}^{x+c_{l_2}t}\psi_1(\xi)d\xi\right)+$$
$$\frac{1}{c_{l_1}}\int_0^t d\tau\int_{x-c_{l_1}t+c_{l_1}\tau}^{x+c_{l_1}t-c_{l_1}\tau}F_1(\tau,\xi)\,d\xi, \tag{14}$$

$$u_2(t,x)=\frac{a_{21}}{2\left(c_{l_1}^2-c_{l_2}^2\right)}\left(\varphi_1(x+c_{l_1}t)+\varphi_1(x-c_{l_1}t)+\frac{1}{c_{l_1}}\int_{x-c_{l_1}t}^{x+c_{l_1}t}\psi_1(\xi)d\xi\right)-$$

$$\frac{c_{l_2}^2 - a_{22}}{2\left(c_{l_1}^2 - c_{l_2}^2\right)}\left(\varphi_2(x + c_{l_1}t) + \varphi_2(x - c_{l_1}t) - \frac{1}{c_{l_1}}\int_{x-c_{l_1}t}^{x+c_{l_1}t} \psi_2(\xi)d\xi\right) +$$

$$\frac{c_{l_1}^2 - a_{22}}{2\left(c_{l_1}^2 - c_{l_2}^2\right)}\left(\varphi_2(x + c_{l_2}t) + \varphi_2(x - c_{l_2}t) + \frac{1}{c_{l_2}}\int_{x-c_{l_2}t}^{x+c_{l_2}t} \psi_2(\xi)d\xi\right) -$$

$$\frac{a_{21}}{2\left(c_{l_1}^2 - c_{l_2}^2\right)}\left(\varphi_1(x + c_{l_2}t) + \varphi_1(x - c_{l_2}t) + \frac{1}{c_{l_2}}\int_{x-c_{l_2}t}^{x+c_{l_2}t} \psi_1(\xi)d\xi\right) +$$

$$\frac{1}{c_{l_2}}\int_0^t d\tau \int_{x-c_{l_2}t+c_{l_2}\tau}^{x+c_{l_2}t-c_{l_2}\tau} F_2(\tau,\xi)\, d\xi. \qquad (15)$$

The classical solution to problem (1) – (4) is called the function

$$u_1, u_2 \in C^2(R_+ \times R) \cap C^1(\bar{R}_+ \times R),$$

where $\boldsymbol{C^k}$ is the space of $\boldsymbol{k}$ times continuously differentiable functions.

From formulas (14) and (15) the following follows:

Theorem. *Let* $\boldsymbol{\varphi_1, \varphi_2 \in C^2(R)}$ $\boldsymbol{\psi_1, \psi_2 \in C^1(R)}$, $\boldsymbol{F_1, F_2 \in C^1(\bar{R}_+ \times R)}$ *then there is a unique classical solution to problem (1) – (4), which is given by the d'Alembert formula, and the following estimate is valid:*

$$\|u_1\|_{C([0,T]\times R)} + \|u_2\|_{C([0,T]\times R)} \le$$

$$C(\|\varphi_1\|_{C(R)} + \|\varphi_2\|_{C(R)} + \|\psi_1\|_{C(R)} + \|\psi_2\|_{C(R)} + \|F_1\|_{C([0,T]\times R)} + \|F_2\|_{C([0,T]\times R)}),$$

where $\boldsymbol{0 < C}$ *is some constant depending only on* $\boldsymbol{T}$.

Proof. It is easy to see that for $\boldsymbol{\varphi_1, \varphi_2 \in C^2(R)}$, $\boldsymbol{\psi_1, \psi_2 \in C^1(R)}$, $\boldsymbol{F_1, F_2 \in C^1(\bar{R}_+ \times R)}$ the functions defined by the d'Alembert formulas (14), (15) give us the classical solution to problem (1)-(4).

The uniqueness of the classical solution follows immediately from the uniqueness of the integrals when obtaining the d'Alembert formula.

IV. CONCLUSION

Thus, we proved that the Cauchy problem is posed for the system (1), (2) described by three elasticity parameters. The formulas built for the solution of this problem can be used to solve the problems of elastic wave theory in numerical metods.

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