Stabilization of a Membrane with Optimization of Placements of Measurement Points and Oscillation Dampers

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***Abstract*—The purpose of this work is providing an approach to the synthesis of control with lumped sources in distributed systems with feedback. In the problem presented here concerning the damping of the oscillations in a membrane using stabilizers, the following parameters are optimized: 1) the locations of the stabilizers; 2) the location of the points of measurement of the state of the membrane; 3) linear feedback parameters, which determine the relationship between the measurements of the state of the membrane and the modes of control of the stabilizers. Gradient formulas for the minimizing function are obtained with respect to optimizing parameters.**

Keywords—thin membrane, oscillation, synthesis of control, lumped source, the neighborhood of points of control, gradient projection method

I. INTRODUCTION

It is known that, compared with control of objects with lumped parameters [1]-[3], problems of optimal control with feedback of objects with distributed parameters are less studied by scientists [4]-[8] . There are many reasons for this. The first is the complexity of technical implementation of such systems which demands that current information on a state of an object (process) at all of its points be obtained. Secondly, there are complexities related both to the solution of problems of the structural and parametric identification of mathematical models of controlled objects, and the development of effective numerical methods, algorithms for solving the relevant mathematical problems.

This work presents an approach to the effect of the synthesis of control on the process of the stabilization of oscillation in a thin homogeneous membrane. It is assumed that the oscillations arise as a result of the lumped influences on the membrane at the initial moment of time. The values of stabilizing effects on the membrane of stabilizers which are located at various points are defined by depending on the measured state of the membrane in the neighborhood of the measurement points.

For synthesis of the control, it is proposed to use linear feedback with the accounting of results of the current measurements. The following are optimized in the considered problem: 1) parameters of the linear feedback: 2) coordinates of location of stabilizers on a membrane: 3) coordinates of measurement points of a state of a membrane. The gradient components of the minimizing functional on the space of optimized parameters are obtained.

II. STATEMENT OF THE PROBLEM

We study the problem of damping of transverse oscillation of a given thin homogenous membrane with a fixed boundary. It is assumed that oscillations arise as a result of simultaneous actions of external sources at the initial moment of time of the neighborhood of some points θ^γ , $\gamma = 1, \dots, L$ of the membrane. Oscillations are damped by stabilizers (dampers) with local effects at the neighborhood of membrane's points, η^i , $i = 1, \dots, N_c$. For the formulation of the modes of operation of the stabilizers, there are used data from the sensors when measuring of the current values of displacement at the neighborhood of the points ξ^j, $j = 1, \dots, N_o$.

The process for $t > 0$ can be described by the following initial-boundary value problem [9],[10]:

$$u_{tt}(x,t) = a^2 \Delta u(x,t) - \lambda u_t(x,t) + \tag{1}$$

$$+ \sum_{s=1}^{N_t} \delta_{\varepsilon_t}(t;\tau_s) \sum_{i=1}^{N_c} \vartheta_s^i \delta_{\varepsilon_x}(x;\eta^i), \;\; x = (x_1, x_2) \in \Omega \subset \mathrm{R}^2,$$

$$u(x,0) = 0, \;\; u_t(x,0) = \sum_{\gamma=1}^{L} q^\gamma \delta_{\varepsilon_x}(x;\theta^\gamma), \;\; x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \;\; x \in \Gamma. \tag{3}$$

Here: $u(x,t)$ is the function which is determine value of a displacement of the membrane at the a point of $x \in \Omega$ in the time of t in the process of its oscillation; $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$; $a^2 > 0$, $\lambda \ge 0$ are the given constants defined by physical properties of a membrane and environment in which it is; Γ is almost everywhere smooth border of area of Ω, which is occupied by the membrane; q^γ is intensity of γ^{th} an external source, the lumped at the point $\theta^\gamma = (\theta_1^\gamma, \theta_2^\theta) \in \Omega$ of membrane where $\gamma = 1, \dots, L$, $\vartheta = (\vartheta_1^1, \dots, \vartheta_1^{N_c}, \dots, \vartheta_{N_t}^1, \dots, \vartheta_{N_t}^{N_c}) \in \mathrm{R}^{N_t N_c}$ is the vector determining the effects of stabilizers at the neighborhood of the points $\eta = (\eta_1^i, \eta_2^i) \in \Omega$, $i = 1, \dots, N_c$, $\tau = (\tau_1, \dots, \tau_{N_t})$ is the given time moment in which at the neighborhood take place effect of the stabilizers, where $\tau_s > \tau_{s-1} > 0$, $s = 1, \dots, N_t$, $\tau_0 = 0$, $\tau_{N_t} = T_f$; N_t is number of impulses; T_f is duration of time of observation of the process.

Continuously differentiable function $\delta_{\varepsilon_x}(x;\hat\eta)$, for a all $x \in \Omega$ determines the distribution intensity of the source at the $\mathcal{O}_{\varepsilon_x}(\hat\eta)$ neighborhood of near the point $\hat\eta \in \Omega$. It satisfies the following properties:

$$\delta_{\varepsilon_x}(x;\hat\eta)\begin{cases}\neq 0 & if \quad x\in\mathcal{O}_{\varepsilon_x}(\hat\eta),\\ = 0 & if \quad x\notin\mathcal{O}_{\varepsilon_x}(\hat\eta),\end{cases}$$

$$\iint_{\mathcal{O}_{\varepsilon_x}(\hat\eta)}\delta_{\varepsilon_x}(x;\hat\eta)\,dx = 1,\ \ \hat\eta\in\Omega.$$

Function $\delta_{\varepsilon_t}(t;\hat\tau)$ is continuous for all $t\in[0,T_f]$ and:

$$\delta_{\varepsilon_t}(x;\hat\tau)\begin{cases}\neq 0 & if \quad t\in\mathcal{O}_{\varepsilon_t}(\hat\tau),\\ = 0 & if \quad t\notin\mathcal{O}_{\varepsilon_t}(\hat\tau),\end{cases}$$

$$\int_{\mathcal{O}_{\varepsilon_t}(\hat\tau)}\delta_{\varepsilon_t}(t;\hat\tau)dt = 1.$$

Let the values of the powers of the sources of oscillations q^γ and their locations θ^γ, $\gamma=1,\dots,L$, are not known exactly. There are given sets of possible values of q^γ:

$$Q^\gamma=\left\{q\in \mathrm{R}:\ \underline{q^\gamma}\le q\le\overline{q^\gamma}\right\},\ \ \gamma=1,\dots,L, \tag{4}$$
$$Q=Q^1\times\cdots\times Q^L,$$

and the density functions of the distribution of their values $\rho_{Q^\gamma}(q)\ge 0$, $\underline{q^\gamma}\le q\le\overline{q^\gamma}$, such that

$$\int_{Q^\gamma}\rho_{Q^\gamma}(q)\,dq = 1,\ \ \gamma=1,\dots,L.$$

Here the points θ^γ of the possible location of sources of external force are determined by the given domains

$$\Theta^\gamma\subset\Omega,\ \ \gamma=1,\dots,L,\ \ \Theta=\Theta^1\times\cdots\times\Theta^L, \tag{5}$$

and with a distribution density function $\rho_{\Theta^\nu}(\theta)\ge 0$ such that

$$\iint_{\Theta^\gamma}\rho_{\Theta^\gamma}(\theta)d\theta = 1,\ \ \gamma=1,2,\dots,L.$$

The values of ϑ_s^i, which determines the control of the powers of effect and their locations η^i is called the optimized parameters of the control of damping of oscillation, it satisfies the following constraints:

$$\underline{\vartheta^i}\le\vartheta_s^i\le\overline{\vartheta^i},\ \ i=1,\dots,N_c,\ \ s=1,\dots,N_t, \tag{6}$$

$$\eta^i\in\mathcal{O}_{\varepsilon_x}(\eta^i)\subset\Omega_c^i\subset\Omega,\ \ i=1,\dots,N_c. \tag{7}$$

In (7) Ω_c^i is the given closed subset in which, the stabilizers can be installed on it, $\underline{\vartheta^i}$, $\overline{\vartheta^i}$ are given where $i=1,2,\dots,N_c$.

To formulate the considered problem, we can state that, consider of control process of damping oscillation of membrane for a given time T_f for which it must be to determine the values of the effects of power for damping of osculation ϑ and their locations η that satisfy the above conditions and minimize the following functional:

$$\mathcal{J}(\vartheta,\eta)=\int_Q\iint_\Theta I(\vartheta,\eta;q,\theta)\rho_\Theta(\theta)\rho_Q(q)d\theta dq\,, \tag{8}$$

$$I(\vartheta,\eta;q,\theta)=\int_{T_f}^{T_1}\iint_\Omega\mu(x)[u(x,t;\vartheta,\eta,q,\theta)]^2\,dx\,dt+ \tag{9}$$
$$+\varepsilon_1\left\|\vartheta-\hat\vartheta\right\|^2_{\mathrm{R}^{N_tN_c}}+\varepsilon_2\|\eta-\hat\eta\|^2_{\mathrm{R}^{2N_c}},$$
$$q=(q^1,\dots,q^L),\ \ \rho_Q(q)=\rho_{Q^1}(q^1)\times\cdots\times\ \rho_{Q^L}(q^L),$$
$$\theta=(\theta^1,\dots,\theta^L),\ \ \rho_\Theta(\theta)=\rho_{\Theta^1}(\theta^1)\times\cdots\times\rho_{\Theta^L}(\theta^L).$$

Here the function $u(x,t)=u(x,t;\vartheta,\eta,q,\theta)$ is the solution of the initial-boundary value problem (1)–(3) of given external effect with the power q^γ at the initial moment of time, $\gamma=1,\dots,L$, and damping modes ϑ; $\mu(x)\ge 0$ is a weight function that determines the value of damping of oscillation at the point $x\in\Omega$ of the membrane. The second term in (8), (9) respond to regularize of the functional, ε_1, ε_2, $\hat\vartheta\in\mathrm{R}^{N_tN_c}$, $\hat\eta\in\mathrm{R}^{2N_c}$ are regularization parameters.

The value of ΔT defines time duration of the interval $[T_f,T_1]$, where $[T_f,T_1]$, $T_1=T_f+\Delta T$, in which it must be observe steady-state of the membrane on this interval.

Suppose that at the points $\xi^j=(\xi_1^j,\xi_2^j)\in\Omega$, $j=1,\dots,N_o$, of the membrane are installed sensors that measure the integral values of the displacement at the neighborhood of these points at times $\tau_s\in(0,T_f]$, $s=1,\dots,N_t$:

$$\hat u_s^j=\int_{\mathcal{O}_\varepsilon(\tau_s)}\iint_{\mathcal{O}_\varepsilon(\xi^j)}u(x,t)\delta_{\varepsilon_x}(x;\xi^j)\delta_{\varepsilon_t}(t;\tau_s)\,dx\,dt, \tag{10}$$
$$j=1,\dots,N_o,\ \ s=1,\dots,N_t,$$

and it is possible to promptly define admissible stabilization modes ϑ_s^i, $i=1,\dots,N_c$, $s=1,\dots,N_t$ based on the results of these measurements.

Here $\delta_{\varepsilon_x}(x;\xi^j)$, $\delta_{\varepsilon_t}(t;\tau_s)$ play a role of the weight functions defining a contribution of value of a condition of a membrane in point $x\in\mathcal{O}_{\varepsilon_x}(\xi^j)$, $t\in\mathcal{O}_{\varepsilon_t}(\tau_s)$ in general on an observed value of $\hat u_s^j$.

To assign current values of stabilizer modes (10), we use the following function, which determines the feedback of control effect with the state of the membrane at the neighborhood of observation points:

$$\vartheta_s^i=\sum_{j=1}^{N_o}k_j^i\left[\int_{\mathcal{O}_\varepsilon(\tau_s)}\iint_{\mathcal{O}_\varepsilon(\xi^j)}u(x,t)\delta_{\varepsilon_x}(x;\xi^j)\delta_{\varepsilon_t}(t;\tau_s)\,dx\,dt-z_j^i\right],$$
$$i=1,\dots,N_c,\ \ s=1,\dots,N_t. \tag{11}$$

Here $k=((k_j^i))$ are reinforcing coefficient; $z=((z_j^i))$, z_j^i are the nominal values of the displacement at the point ξ^j with respect to the stabilizer which is installed at the point η^i, $i=1,\dots,N_c$, $j=1,\dots,N_o$; k and z are optimized feedback parameters.

Substituting formula (11) into equation (1), we obtain:

$$u_{tt}(x,t)=a^2\mathcal{L}u(x,t)-\lambda u_t(x,t)+ \tag{12}$$
$$+\sum_{s=1}^{N_t}\delta_{\varepsilon_t}(t;\tau_s)\sum_{i=1}^{N_c}\delta_{\varepsilon_x}(x;\eta^i)\times$$
$$\times\sum_{j=1}^{N_o}k_j^i\left[\int_{\mathcal{O}_\varepsilon(\tau_s)}\iint_{\mathcal{O}_\varepsilon(\xi^j)}u(\tilde x,\tilde t)\delta_{\varepsilon_x}(\tilde x;\xi^j)\delta_{\varepsilon_t}(\tilde t;\tau_s)\,d\tilde x\,d\tilde t-z_j^i\right],$$

Due to the involving values of the unknown functions at the neighborhood of some points of the phase space in the right side of differential equations, such equations are called loaded differential equations [11]-[14].

Let devices for measuring the state of a membrane, based on technical and technological considerations, be installed in some of its specified sub-regions:

$$\xi^j\in\mathcal{O}_{\varepsilon_x}(\xi^j)\subset\Omega_o^j\subset\Omega,\ \ j=1,\dots,N_o, \tag{13}$$

Main aim of this article is synthesis of the controlling parameters of stabilization of process of a transverse oscillation of a membrane. The problem consists to finding of optimal values of parameters of a feedback control of $k\in\mathrm{R}^{N_cN_o}$, $z\in\mathrm{R}^{N_cN_o}$, locations of measurement points ξ and stabilization η at which restrictions (6), (7), (12), (2), (3) are satisfied. The total dimension of the finite-dimensional vector of the synthesized parameters, which we denote by $y = (k, z, \xi, \eta)$ is equal $N = 2(N_c N_o + N_c + N_o)$ i.e., $y \in \mathrm{R}^N$.

We will write the quality criteria of controlling parameter, which is defined by a functional (8), as:

$$\mathcal{J}(y) = \int_Q \iint_\Theta I(y; q, \theta)\rho_\Theta(\theta)\rho_Q(q)\, d\theta\, dq, \tag{14}$$

$$I(y; q, \theta) = \int_{T_f}^{T_1} \iint_\Omega \mu(x)[u(x, t; y, q, \theta)]^2\, dx\, dt + \tag{15}$$

$$+\varepsilon_1\|k - \hat{k}\|^2_{\mathrm{R}^{N_c N_o}} + \varepsilon_2\|z - \hat{z}\|^2_{\mathrm{R}^{N_c N_o}} + \varepsilon_3\|\xi - \hat{\xi}\|^2_{\mathrm{R}^{2N_o}} + \varepsilon_4\|\eta - \hat{\eta}\|^2_{\mathrm{R}^{2N_c}}.$$

Here $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, $\varepsilon_i \geq 0$, $i = 1, \ldots, 4$, $\hat{k} \in \mathrm{R}^{N_c N_o}$, $\hat{z} \in \mathrm{R}^{N_c N_o}$, $\hat{\xi} \in \mathrm{R}^{2N_o}$, $\hat{\eta} \in \mathrm{R}^{2N_c}$ are regularization parameters.

The obtained problem of synthesis the control of stabilization of the oscillations (12), (2), (3), (4)–(6), (14), (15) belongs to the class of parametric optimal control problems for distributed systems.

III. Approach and Formulas for the Solution of the Problem of Synthesis of Control Stabilization of Process

First of all, we consider the constraints (6), (7), (13). We assume that the domain Ω has a simple structure as (such as rectangle, circle, ellipse, etc.), and therefore the projection operator of the points of the space R^2 on this region has a constructive character.

Taking into account (6), (11) and the notation:

$$g_i^0(\tau_s; y) = \frac{\overline{\vartheta_i} + \underline{\vartheta_i}}{2} - \sum_{j=1}^{N_o} k_j^i \left[\int_{O_\varepsilon(\tau_s)} \iint_{O_\varepsilon(\xi^j)} u(x, t)\delta_{\varepsilon_x}(x; \xi^j)\delta_{\varepsilon_t}(t; \tau_s)\, dx\, dt - z_j^i \right],$$
$$i = 1, \ldots, N_c, \quad s = 1, \ldots, N_t,$$

the constraints take the following compact form:

$$g_i(\tau_s; y) = |g_i^0(\tau_s; y)| - \frac{\overline{\vartheta_i} - \underline{\vartheta_i}}{2} \leq 0, \tag{16}$$
$$i = 1, \ldots, N_c, \quad s = 1, \ldots, N_t.$$

Considering the constraints (16), let us use external penalty method [15],[16] in the problem of optimizing the feedback parameters y. Adding penalty term to the integrand function of the objective functional (15), we can write:

$$\mathcal{J}_r(y) = \int_Q \iint_\Theta \tilde{I}_r(y; q, \theta)\rho_\Theta(\theta)\rho_Q(q)\, d\theta\, dq, \tag{17}$$

$$\tilde{I}_r(y; q, \theta) = I(y; q, \theta) + rG(y). \tag{18}$$

Here following notation are used:

$$G(y) = \sum_{s=1}^{N_t} \sum_{i=1}^{N_c} [g_i^+(\tau_s; y)]^2,$$

$$g_i^+(\tau_s; y) = \begin{cases} 0, & g_i(\tau_s; y) \leq 0, \\ g_i(\tau_s; y), & g_i(\tau_s; y) > 0. \end{cases}$$

In (18), the parameter $r > 0$ is called penalty coefficient, and for convenience of numerical calculations it is necessary that putting r approaches to $+\infty$.

To take into account the constraints (7), (13), we use the projection operators on Ω_c^i, Ω_o^j, $i = 1, \ldots, N_c$, $j = 1, \ldots, N_o$. In general, for the numerical solution of the problem of synthesis of parameters y, we use the gradient projection method of the penalty functional. The iteration procedure for constructing a minimizing sequence has the following form [15]:

$$y^{m+1} = \mathcal{P}_{(7),(13)}[y^m - \alpha_m \operatorname{grad} \mathcal{J}_r(y^m)], \tag{19}$$
$$m = 0, 1, 2, \ldots.$$

Here $\operatorname{grad} \mathcal{J}_r(y)$ is the gradient of the functional (17), evaluated at the point $y \in \mathrm{R}^N$ with a given penalty coefficient r; $\mathcal{P}_{(7),(13)}[\cdot]$ is the projection operator of components of the vector y (more precisely, its components ξ and η) on the sub-region Ω_c^i и Ω_o^j, $i = 1, \ldots, N_c$, $j = 1, \ldots, N_o$; α_m is the step size in the direction of the penalty function designed for the restrictions (7), (13) of the anti-gradient at which the condition $\mathcal{J}_r(y^{m+1}) \leq \mathcal{J}_r(y^m)$ [15] should be fulfilled.

To apply the procedure (19), we formulate a theorem in which gives us opportunity to write explicit form for the components of the gradient of the functional $\mathcal{J}_r(y)$. In here we use the characteristic function $\chi_{t \in [T, T_1]}(t)$ which is equal to zero when $t \notin [T_f, T_1]$ and is equal to one when $t \in [T_f, T_1]$.

Theorem. *For the component of gradient of the functional (14), (15), corresponding to the parameters* $y = (k, z, \xi, \eta) \in \mathrm{R}^N$ *for the continuous linear feedback (10), (11), has the following form:*

$$\frac{\partial \mathcal{J}_r(y)}{\partial k_j^i} = \int_Q \int_\Theta \Big\{ -\sum_{s=1}^{N_t} \left[\hat{\psi}_s^i + 2r g_i^+(\tau_s; y) \operatorname{sgn}\left(g_i^0(\tau_s; y)\right) \right] \times [\hat{u}_s^j - z_j^i] + 2\varepsilon_1(k_j^i - \hat{k}_j^i) \Big\} \rho_\Theta(\theta)\rho_Q(q)\, d\theta\, dq, \tag{20}$$

$$\frac{\partial \mathcal{J}_r(y)}{\partial z_j^i} = \int_Q \iint_\Theta \left\{ \sum_{s=1}^{N_t} \left[\hat{\psi}_s^i + 2r g_i^+(\tau_s; y) \operatorname{sgn}\left(g_i^0(\tau_s; y)\right) \right] \times k_j^i + 2\varepsilon_2(z_j^i - \hat{z}_j^i) \right\} \rho_\Theta(\theta)\rho_Q(q)\, d\theta\, dq, \tag{21}$$

$$\frac{\partial \mathcal{J}_r(y)}{\partial \xi_\gamma^j} = \int_Q \iint_\Theta \left\{ -\sum_{s=1}^{N_t} \sum_{i=1}^{N_c} \left[\hat{\psi}_s^i + 2r g_i^+(\tau_s; y) \operatorname{sgn}\left(g_i^0(\tau_s; y)\right) \right] \times k_j^i \hat{u}_{s,x}^j + 2\varepsilon_3(\xi_\gamma^j - \hat{\xi}_\gamma^j) \right\} \rho_\Theta(\theta)\rho_Q(q)\, d\theta\, dq, \tag{22}$$

$$\frac{\partial \mathcal{J}_r(y)}{\partial \eta_\gamma^i} = \int_Q \iint_\Theta \left\{ -\sum_{s=1}^{N_t} \hat{\psi}_{s,x}^i \sum_{j=1}^{N_o} k^{ij}[\hat{u}_s^j - z^{ij}] + 2\varepsilon_4(\eta_\gamma^i - \hat{\eta}_\gamma^i) \right\} \times \rho_\Theta(\theta)\rho_Q(q)\, d\theta\, dq, \tag{23}$$

where $\hat{\psi}_s^i = \int_{O_{\varepsilon_t}(\tau_s)} \iint_{O_{\varepsilon_x}(\eta^i)} \psi(x, t)\delta_{\varepsilon_x}(x; \eta^i)\delta_{\varepsilon_t}(t; \tau_s)\, dx$. $i = 1, \ldots, N_c$, $j = 1, \ldots, N_o$, $s = 1, \ldots, N_t$ $\gamma = 1, 2$. *The function* $\psi(x, t)$ *is called solution of following adjoint inital-boundary value problem:*

$$\psi_{tt}(x, t) = a^2 \Delta\psi(x, t) + \lambda\psi_t(x, t) \tag{24}$$
$$-2u(x, t; y, q, \theta)\chi_{t \in [T, T_1]}(t) + \sum_{s=1}^{N_t} \delta_{\varepsilon_t}(t; \tau_s) \sum_{j=1}^{N_o} \delta_{\varepsilon_x}(x; \xi^j) \times \sum_{i=1}^{N_c} k_j^i \left[\hat{\psi}_s^i + 2r g_i^+(\tau_s; y) \operatorname{sgn}\left(g_i^0(\tau_s; y)\right) \right],$$

$$x \in \Omega, t \in [0, T_1],$$
$$\psi(x, T_1) = 0, \ \ \psi_t(x, T_1) = 0, \ \ x \in \Omega, \tag{25}$$
$$\psi(x, t) = 0, \ \ x \in \Gamma, \ \ t \in [0, T_1]. \tag{26}$$

IV. Results of Numerical Experiments

We present the results of numerical experiments obtained by solving the problem in (1)–(4) with the following data values, which is involved in the formulation of the problem:

$$\Omega = \{x \in \mathrm{R}^2\text{:} \ \ 0 \le x_i \le 1, \ \ i = 1{,}2\}, \ \ T_f = 3, \ \ \Delta T = 0.3,$$
$$a = 1, \ \ \lambda = 0.001, \ \ N_c = 2, \ \ N_o = 3,$$
$$\underline{\vartheta_i} = -0.05, \ \ \overline{\vartheta_i} = 0.05, \ \ i = 1, \dots, N_c, \ \ L = 2,$$
$$N_t = 10, \ \ \tau_s = 0.3s, \ \ s = 1, \dots, N_t,$$
$$\Omega_o^1 = \{x \in \Omega\text{:} \ \ 0.12 \le x_1 \le 0.32, \ \ 0.28 \le x_2 \le 0.48\},$$
$$\Omega_o^2 = \{x \in \Omega\text{:} \ \ 0.71 \le x_1 \le 0.89, \ \ 0.05 \le x_2 \le 0.22\},$$
$$\Omega_o^3 = \{x \in \Omega\text{:} \ \ 0.44 \le x_1 \le 0.64, \ \ 0.45 \le x_2 \le 0.63\},$$
$$\Omega_c^1 = \{x \in \Omega\text{:} \ \ 0.58 \le x_1 \le 0.78, \ \ 0.25 \le x_2 \le 0.42\},$$
$$\Omega_c^2 = \{x \in \Omega\text{:} \ \ 0.19 \le x_1 \le 0.39, \ \ 0.59 \le x_2 \le 0.78\},$$
$$q_1 \in Q^1 = \{0.050; \ 0.051; \ 0.052; \ 0.053\},$$
$$q_2 \in Q^2 = \{0.049; \ 0.050; \ 0.051; \ 0.052\},$$
$$\theta^1 \in \Theta^1 = \{x \in \Omega\text{:} \ \ 0.22 \le x_1 \le 0.34, \ \ 0.20 \le x_2 \le 0.38\},$$
$$\theta^2 \in \Theta^2 = \{x \in \Omega\text{:} \ \ 0.65 \le x_1 \le 0.78, \ \ 0.68 \le x_2 \le 0.81\}.$$

The values of the powers of external effect have uniform distributions in Q^1 and Q^2, and their possible impact points are uniformly distributed in the given admissible regions Θ^1 and Θ^2. The sets of possible allocation points Θ^γ and the values of the powers of external impact Q^γ are approximated by the following discrete sets of points:

$$\Theta^1 = \{(0.26; 0.28), (0.27; 0.29), (0.25; 0.25), (0.32; 0.35)\},$$
$$\Theta^2 = \{(0.75; 0.75), (0.70; 0.78), (0.75; 0.79), (0.76{,}0.80)\}.$$

Each of the discrete sets contains four elements, and the probabilities to obtaining these values, considering the uniformly probability for continuous sets, are equal to

$$p_{Q^1}^{\nu} = p_{Q^2}^{\nu} = 0.25, \ \ \nu = 1, \dots, 4,$$
$$p_{\Theta^1}^{\nu} = p_{\Theta^2}^{\nu} = 0.25, \ \ \nu = 1, \dots, 4.$$

In this case, we will take the following functional

$$\mathcal{J}_r(y) = \frac{1}{256} \sum_{i_1=1}^{4} \sum_{i_2=1}^{4} \sum_{j_1=1}^{4} \sum_{j_2=1}^{4} \tilde{I}_r(y; q^{i_1}, q^{i_2}, \theta^{j_1}, \theta^{j_2}).$$

The optimized vector of feedback control parameters with the above selected parameters has dimension $N = 22$. Set of possible locations of the stabilizers Ω_c^i and the sensors Ω_o^j, as is useful in practical applications, are located inside the membrane, i.e. they cannot be located on the boundary of Γ.

We describe the general scheme for implementing the iterative procedure (19) using gradient formulas (20)—(23) to minimize the functional (17) using the penalty function and gradient projection methods.

Considering that the admissible regions of the parameters ξ, η are rectangular, the projection operators for these regions are obvious and have a simple form [15].

At each value of the penalty coefficient r, the functional is regularized by using well-known schemes [15],[16]. In this case, the regularization parameters were changed three times, namely, at the initial value of $\varepsilon = 0.1$, it decreased by 5 times, and the obtained optimal value $\hat{y}$ at the previous step was assigned by y. The initial value of the penalty coefficient is putting equal to 5, which at each subsequent stage increased 5 times. These stages were carried out until at two successive stages the value of the main functional of the problem (17), (18) differs from each other more than by 0.005.

To solve two-dimensional direct and conjugate initial-boundary value problems (12), (2), (3) and (24)–(26), the Alternating-Direction method [17] is used, leading to the solution of one-dimensional loaded problems. To solve the loaded initial-boundary value problems, an implicit finite-difference approximation scheme is used, which was studied in [12]. To solve the finite-difference approximated initial-boundary value problems, the numerical methods proposed in [13] were used. The steps for approximation with respect to the spatial variable were chosen equal $h_{x_1} = h_{x_2} = 0.01$ and the time variable $h_t = 0.005$.

The function $\delta_{\varepsilon_x}(x; 0)$ was defined as the following Gaussian-type function [1],[18].

Computer experiments are carried out to observe the process of damping of oscillations at optimal values of the synthesized feedback parameters with the assuming that the measurements are made with errors (noises), namely:

$$\tilde{u}_s^j = \int\limits_{O_\varepsilon(\tau_s)} \iint\limits_{O_\varepsilon(\xi^j)} u(x,t)\left[1 + \chi^j(t)\right] \delta_{\varepsilon_x}(x; \xi^j) \delta_{\varepsilon_t}(t; \tau_s)\, dx\, dt,$$
$$j = 1, \dots, N_o, \ \ s = 1, \dots, N_t.$$

Here $\chi^j(t)$, for every t, is a random variable uniformly distributed on the interval $[-\zeta; \zeta]$. In the experiments, the values of ζ were chosen to be 0.01; 0.02; 0.05, which corresponded to a measurement error of 1%, 2% and 5% of the value of the measured value.

An important indicator of the quality of control of damping process with the feedback parameters y is the function:

$$E(T; y^*) = \int\limits_{Q} \iint\limits_{\Theta} \left[\int\limits_{T}^{T+\Delta T} \iint\limits_{\Omega} \mu(x) [u(x, t; y^*, q, \theta)]^2\, dx\, dt \right]$$
$$\times\, \rho_\Theta(\theta) \rho_Q(q)\, d\theta\, dq, \ \ T \ge 0,$$

numerically characterizing the result of process control on average for all possible values of external effect. Figure 1 shows the graphs of the function $E(T; y^*)$ which is obtained with optimal feedback parameters y^* and $T_f = 3$ at error levels in measurements are equal to 0% (without error), 1%, 2% and 5%. From these graphs it can be seen that the quality of the control of damping process corresponds to the error values of the process state measurements. In fig. 2 is a graph of the function $u(x, t) = u(x, t; y^*, q, \theta)$, $x \in \Omega$ which determines the state of the membrane at the time of the end of the control process at $t = T_f = 3$ and $q_1 = 0.52$, $q_2 = 0.49$, $\theta_1 = (0.25, 0.25)$, $\theta_2 = (0.75, 0.75)$.

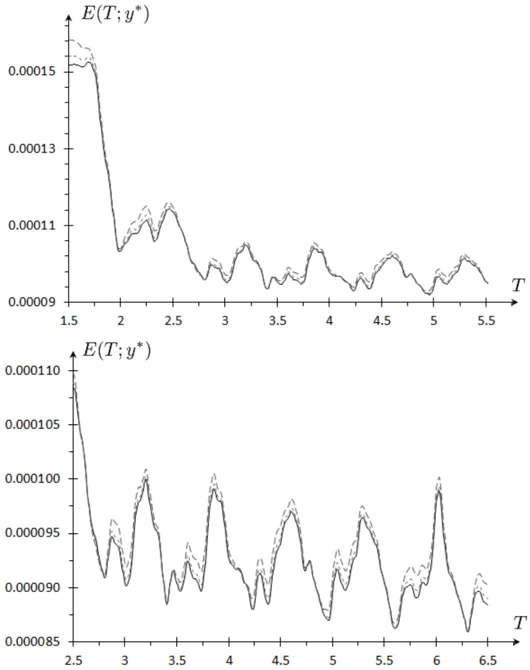

Fig. 1. Graphs of function $E(T, y^*)$ at different errors of $u(x, T)$ at $T = 3$.

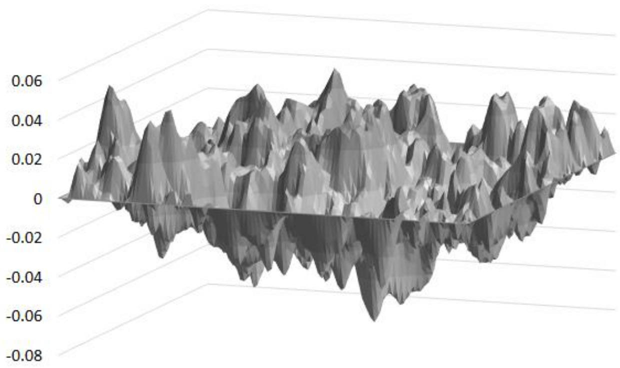

Fig. 2. Graphs of function $u(x, t)$ at $T = 3$.

Conclusion

The purposes of this paper is an approach to the optimal synthesis of lumped control effect in systems with distributed parameters. As an example, we consider the control of lumped stabilizers in damping oscillation of membrane. The modes of operation of the stabilizers are determined by a linear dependence on measurements of the state of the membrane in the neighborhood of the measuring points. In problem, it is optimized: 1) the parameters of linear feedback, which determines the modes of operation of stabilizers; 2) locations of stabilizers; 3) the location of the points of measurement of the state of the membrane.

The considered problem is reduced to a parametric optimal control of a system with distributed parameters. Gradient formula for the functional of the considering problem in the space of synthesized parameters are obtained. The formulas made it possible to use efficient first-order optimization methods for the numerical solution of the synthesis problem.

In paper, it presents the results of numerical experiments and analyzes the effect of measurement errors on the stabilization process of the membrane.

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