Asymptotic Properties of Survival Probability in Discrete-Time Subcritical Galton-Watson Systems

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***Abstract*—In this paper, Kolmogorov's famous theorem is generalized to the subcritical case, known from the discrete-time subcritical Galton-Watson branching processes theory. In this theorem, the Kolmogorov constant for the subcritical state is defined, and its exact form is determined. The obtained results shed light on several important classical results in the subcritical random branching processes theory. In particular, using our results, local limit theorems are derived for subcritical random branching theories.**

***Keywords*—Generating function, Kolmogorov constant, extinction probability, survival probability.**

I. INTRODUCTION

In this paper, we consider the Galton-Watson (G-W) process. In this case, the age distribution of the particles is given by a unit-step function. The process evolves discretely in time as follows. Each particle, independently of the number of particles in the system and its past, will leave $k \in N_0$ offspring with probability $p_k > 0$ at the end of its life, where $N_0 = \{0\} \cup N$ and N denotes the set of natural numbers. Let the number of particles at time n be denoted by $Z(n)$, and we observe that for this sequence, the following equation holds:

$$Z(n+1) = \xi_{n1} + \xi_{n2} + \cdots \xi_{nZ(n)},$$

where ξ_{nk} are independent random variables with the common distribution $P\{\xi_{11} = k\} = p_k$, which are interpreted as the number of offspring of the k-th particle in the n-th generation; [1, pp. 1-2], [10, p. 19]. To exclude trivial cases, we impose the conditions $p_k \neq 1$, and $p_0 + p_1 < 1$. The sequence of offspring $\{Z(n),\ n \in N_0\}$ of the defined process generates a discrete state space, a branching, and a discrete-time homogeneous Markov chain. Its state space is $S_0 = \{0\} \cup S$, $\{0\}$ is an absorbing state, and $S \subset N$ forms the set of connected significant states. The transition probabilities of this chain for all $i, j \in S_0$ and $n \in N_0$ are given by:

$$P_{ij} \coloneqq P\{Z(n+1) = j | Z(n) = i\} = \\ = \sum_{k_1+k_2+\cdots+k_i=j} p_{k1} \cdot p_{k2} \cdot \ldots \cdot p_{ki} \qquad (1)$$

which satisfies the branching property. Conversely, any Markov chain $\{p_k, k \in N_0\}$ that satisfies property (1) is a Galton-Watson (G-W) process with a reproductive law. From the above, it follows that the distribution $\{p_k\}$ of the law completely determines the G-W process. Now, let us consider the transition probability from an arbitrary state $i \in S$ to a state $j \in S_0$ in $n \in N$ steps:

$$P_{ij}(n) \coloneqq P_i\{Z(n) = j\} := P\{Z(n+k) = j | Z(k) = i\},$$

which implies that this probability is independent of the index $k \in N_0$. For the generating function, according to the Kolmogorov-Chepmen equation, the following relation holds:

$$E_i s^{Z(n)} \coloneqq \sum_{j \in S_0} P_{ij}(n) s^j = [F(n; s)]^i, \qquad (2)$$

where $F(n; s) = \sum_{j \in S_0} p_j(n) s^j$ is the generating function with $p_j(n) \coloneqq P_{1j}(n)$ representing the transition probabilities. This function, in turn, is expressed as the n-fold iteration of the generating function:

$$f(s) \coloneqq \sum_{j \in S_0} p_j\, s^j,$$

which satisfies $F(t; s) = f\big(F(t-1; s)\big)$ with $F(1; s) = f(s)$, and is given by the recurrence:

$$F(n+1; s) = f(F(n; s)).$$

The event $\{Z(n) = 0\}$ represents the extinction of the Galton-Watson process at time n. The limit of the probability of the event is $q = lim_{n\to\infty} P\{Z(n) = 0\}$, which is called the extinction probability of a process that starts with a single particle. This probability is equal to the smallest positive solution for $s \in [0,1]$ of the equation $f(s) = s$. Furthermore, for any $r < 1$, the relation $lim_{n\to\infty} F(n; s) = q$ holds uniformly for all $s \in [0,1]$. Let the series $m \coloneqq \sum_{j \in S} j p_j$ converge. Then, $m = f'(1-)$, and this value is equal to the expected number of offspring left by a single particle. According to the relation (2), $EZ(n) = m^n$. Based on this formula, the Galton-Watson process is classified as subcritical, critical, or supercritical if $m < 1, m = 1$, or $m > 1$, respectively. If $m \le 1$, then $q = 1$, and for $m > 1$, $q < 1$ is valid.

We recall the result proven by A. Kolmogorov [3] for the probability of continuation $Q(n) = P\{Z(n) > 0\}$ of a subcritical branching Galton-Watson process $\{Z(n), n \in N\}$.

Kolmogorov's Theorem [3]. *For a subcritical branching Galton-Watson process, if the condition $f''(1-) < \infty$ holds, then the following asymptotic relation is valid*:

$$Q(n) = Km^n\big(1 + o(1)\big), \;\; n \to \infty, \qquad (3)$$

where K is a constant, known as the Kolmogorov constant, which depends on the numerical parameters of the reproductive law $\{p_k, k \in S_0\}$.

According to the formula (3), the probability of survival of the offspring of a single particle tends to zero. On the other hand, using the total probability formula, it is straightforward to calculate that $E[Z(n)|Z(n) > 0] = 1/Q(n)$**.** According to our conditions, the denominator of this relation is positive for all fixed $n \in N$. Thus, there exists a class of positive trajectories such that, for the subcritical case as $n \to \infty$,

$$\frac{m^n}{Q(n)} = \frac{EZ(n)}{P\{Z(n)>0\}} = E[Z(n)|Z(n) > 0] \approx 1/K, \qquad (4)$$

meaning that the expected number of offspring of a single particle converges to $1/K$ as time progresses along positive trajectories.

According to the relation (4) above, the physical meaning of the constant K is that it represents the equivalence coefficient between the expected value of the population size at the current time, $EZ(n) = m^n$, and the survival probability of the population, $Q(n) = P\{Z(n) > 0\}$. Also, in works [13]–[15], limit theorems have been obtained for the survival probability of critical branching processes.

Later, in 1967, A. Nagayev and I. Badalbayev [7] showed that the asymptotic formula (3) also holds for subcritical Galton-Watson processes under the condition weaker than Kolmogorov's, namely:

$$EZ(1)ln^+Z(1) = \textstyle\sum_{k\in S} p_k k ln k < \infty. \qquad (5)$$

II. Results and Discussion

The fact that the exact form of the Kolmogorov constant has not been found is the reason why many limit theorems for random processes that follow subcritical branching laws remain incomplete. This also applies to immigration processes.

Now, we consider the function

$$R(n; s) := q - F(n; s).$$

For a random variable H defined as the extinction moment of the process, where $H := inf\{n: Z(n) = 0\}$, the following equation holds for any $n \in N_0$

$$P\{n < H < \infty|Z(n) = k\} = q^k.$$

From this equation, using the full probability formula $p_k(n) = P\{Z(n) = k\}$, we get the following relationships:

$$\begin{aligned} Q(n) := P\{n < H < \infty\} &= \sum_{k\in S} P\{n < H < \infty, Z(n) = k\} \\ &= \sum_{k\in S} P\{n < H < \infty|Z(n) = k\}p_k(n) \\ &= \sum_{k\in S} p_k(n)q^k = \sum_{k\in S_0} p_k(n)q^k - p_1(n) \\ &= F(n; q) - F(n; 0) = R(n; 0). \end{aligned} \qquad (6)$$

The obtained quantity expresses the probability that the process will continue at time $n \in N$ in the subcritical case. For the supercritical case, it corresponds to the probability of continuation at time $n \in N$ for trajectories that eventually extinguish. At the same time, the above equation leads us to the idea of extending the asymptotic formula (3) to the critical case.

We generalize and prove Kolmogorov's theorem for subcritical branching processes in the following theorem.

Theorem 1. *For subcritical branching G-W processes, if condition (5) is satisfied, the following asymptotic relation holds:*

$$Q(n) = K_q\beta^n\big(1 + o(1)\big), \;\; n \to \infty, \qquad (7)$$

where $\beta = f'(q)$, and K_q is a constant, which we refer to as the generalized Kolmogorov constant. This constant depends on the probability q and the value of the converging series

$$\textstyle\sum_{k\in S} p_k\, q^{k-1} k ln k\, .$$

Proof. We consider the Harris-Sevastyanov transformation $f_q(s) := f(qs)/q$. It is immediately evident that

$$f_q(s) = \textstyle\sum_{k\in S_0} p_k^{(q)}\, s^k,$$

where

$$p_k^{(q)} = p_k q^{k-1} \text{ and } \textstyle\sum_{k\in S_0} p_k^{(q)} = 1.$$

Now, we construct a branching G-W process starting from a single particle with the reproductive law $\{p_k^{(q)}, k \in S_0\}$, and denote it by $\{Z_q(n), n \in N\}$. Then, it follows that $f'_q(1) = \beta < 1$, meaning that $\{Z_q(n)\}$ is a subcritical process. Accordingly, for any $n \in N$, it is not difficult to see that the expected population size $EZ_q(n) = \beta^n$. Next, we calculate the probability $Q_q(n) := P\{Z_q(n) > 0\}$. In fact, this probability is equal to the value of the function

$$R_q(n; s) := 1 - F_q(n; s)$$

at $s = 0$, where $F_q(n; s)$ is the n-th iteration of the function generated by $f_q(s) \coloneqq f(qs)/q$, i.e.

$$F_q(n+1; s) \coloneqq F_q\left(n; f_q(s)\right) = f_q\big(F_q(n; s)\big).$$

This iteration gives us the equality $qR_q(n; s) = R(n; qs)$. Using this equality and equation (6), we obtain the following relation:

$$Q(n) = qR_q(n; 0) = qP\{Z_q(n) > 0\} = qQ_q(n) \qquad (8)$$

In turn, it is easy to see that

$$\sum_{k\in S} p_k^{(q)}\, k\, lnk = \sum_{k\in S} p_k q^{k-1} klnk < \sum_{k\in S} p_k klnk.$$

From the last inequality and the asymptotic formula (3), it follows that when condition (5) holds, the ratio $Q_q(n)/\beta^n$has a finite positive limit as $n \to \infty$. This limit depends on the numerical characteristics of the function $f_q(s)$, and we denote it by K_β. Thus, using formulas (3) and (8), we obtain the following relation:

$$\frac{Q(n)}{\beta^n} = q\,\frac{Q_q(n)}{\beta^n} \longrightarrow qK_\beta,\ \ n \to \infty$$

The asymptotic relation obtained above shows that when condition (5) holds, formula (6) is satisfied with $K_q = qK_\beta$.

The proof of the theorem is complete.

For subcritical G-V processes, we seek the asymptotic expansion of the function $R(n; s) \coloneqq q - F(n; s)$ at all values of sss around the point q, i.e., for

$$s \in U_q[0,1) \coloneqq \{[0, q) \cup (q, 1)\}.$$

In our considerations, we accept the following condition:

$$[m \neq 1\ \ and\ \ m < 1\ \ for\ \ 2b \coloneqq f''(1-) < \infty] \qquad (9)$$

which we refer to as the Kolmogorov condition.

Theorem 2. For subcritical branching G-V processes, if the Kolmogorov condition (9) is satisfied, the following asymptotic relation holds:

$$Q(n) = K_q\beta^n\big(1 + o(1)\big),\quad n \to \infty, \qquad (10)$$

where the generalized Kolmogorov constant is

$$K_q = \frac{q}{1 + q\gamma}$$

with $\gamma = b_q/(\beta - \beta^2)$ and $2b_q \coloneqq f''(q)$. In particular, for the subcritical case, the form of the Kolmogorov constant in formula (3) is

$$K = \frac{1}{1+\gamma}\ ,$$

where $\gamma = b/(m - m^2)$ and $2b \coloneqq f''(1-)$.

III. Conclusion

In this paper, Kolmogorov's [3] famous theorem has been generalized and proven for subcritical Galton-Watson branching processes.

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