New Integral Inequalities for the Trajectory in the Study of the Boltz Type Optimal Control Problem

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***Abstract*: In this study, the Boltz type optimal control problem, which is the most general class among the collected parameter optimal control problems, is studied. Note that depending on the form of the minimized functional, either the terminal control problem, the Lagrangian problem, or the fastest impact problem is a special case of this optimal control problem. In the Boltz-type optimal control problem, new integral inequalities for the trajectory were obtained, which is one of the new mathematical effects obtained in the mathematical theory of optimal control.**

***Keywords*: optimal control problem, Hamilton function, Boltz type optimal control problem, integral inequalities.**

I. INTRODUCTION

Various classes of optimal control problems with lumped parameters have been studied in detail in various aspects [1-3]. This paper presents lumped-parameter optimal control problems described by integral inequalities, which are completely different from optimal control problems described by ordinary differential equations, integral equations, or integro-differential equations. In the work, the obtained integral inequalities for the trajectory in the Boltz type optimal control problem from a broader class of optimal control problems with lumped parameters actually demonstrate optimal control problems described by integral inequalities.

Suppose a controlled object

$$\dot{x}(t) = F(t, x(t), u(t)), t \in [t_0, T], \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

it is illustrated by the Cauchy problem. According to the solution $x(t)$ of equation (1) satisfying the initial condition (2) it is required to find a controller $u(t) \in U_m$ such that

$$B(u) = \int_{t_0}^{T} \Phi(t, x(t), u(t))dt + g(x(T)) \to \min, \tag{3}$$

Usually, a set of piecewise continuous or measurable bounded functions is taken as a class of possible controls U_m .These are the functions $F(t, x(t), u(t))$, $\Phi(t, x(t), u(t))$, $g(x)$ given above. Problem (1)-(3) is called the Boltz problem.

Suppose that controller $u(t) \in U_m$ and corresponding to it function $x(t)$, $t \in [t_0, T]$ is the solution of equation (1) satisfying the initial condition (2). Then the pair $(u(t), x(t))$ is called a process. According to this process of equation

$$\dot{\Psi}(t) = -H_x(t, x(t), \Psi(t), u(t)), \tag{4}$$

let's look at the solution that satisfies the condition

$$\Psi(T) = -\ g_x(x(T)), \tag{5}$$

here

$$H(t, x(t), \Psi(t), u(t)) = \Psi(t)F(t, x(t), u(t)) - \\ -\Phi(t, x(t), u(t))$$

it is called the Hamiltonian function. Problem (4)-(5) is called a supplementary problem to Boltz problem (1)-(3). **Definition.** Certain controller $u^*(t) \in U_m$ and for an arbitrary controller $u(t) \in U_m$ if $B(u^*) \le B(u)$, then controller $u^*(t)$ is called optimal controller due to its functionality $B(u)$. In this case, the pair $(u^*(t), x(t))$ is called an optimal process.

As is known from the mathematical theory of optimal control, that the maximum principle for the Boltz problem (1)-(3) is as follows:

Theorem (Maximum principle) Assume that the $u(t)$ optimal controller, functions $x(t)$ and $\Psi(t)$ respectively, are solutions of problems (1)-(2) and (4)-(5) corresponding to this controller. Then for all points $t \in [t_0, T]$ the maximum condition is paid:

$$H(t, x(t), \Psi(t), u(t)) = \max_{\nu \in U_m} H(t, x(t), \Psi(t), \nu), \tag{6}$$

Note that the maximum condition in (6), can be written as

$$H(t, x(t), \Psi(t), \nu) - H(t, x(t), \Psi(t), u(t)) \le 0, \\ t \in [t_0, T], \nu \in U_m, \tag{7}$$

Inequality (7) is written as an inequality of the maximum condition (6).

The above are known facts related to the Boltz problem (1)-(3).In this work, we study the obtaining of new mathematical effects in the time slice $t \in [t_0, T]$ for the Boltz problem (1)-(3).

II. PROBLEM STATEMENT

The goal is to obtain new integral inequalities for the trajectory $x(t),\ t \in [t_0, T]$ in the Boltz problem (1)-(3). To do this, we will use the following auxiliary fact.

Supporting fact. For positive numbers a and b the following relation is correct:

$$\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}, \tag{8}$$

Formula (8) shows the relationship between the root mean square, numerical mean, geometric mean and harmonic mean for positive numbers a and b .The correctness of this formula can be easily demonstrated using abbreviated multiplication identities.

If in a particular case in formula (8) $a = b$, then $\sqrt{\frac{b^2+b^2}{2}} = \frac{b+b}{2} = \sqrt{bb} = \frac{2bb}{b+b} = b$. As can be seen, the equality condition in formula (8) is satisfied if and only if $a = b$.

Now suppose that and are positive numbers a and b are the vertices of any segment. Without loss of generality, assume that number a is the left end and number b is the right end. Then inequality (8) will be strictly satisfied:

$$\sqrt{\frac{a^2+b^2}{2}} > \frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}, \tag{9}$$

It is clear that the square mean- $\sqrt{\frac{a^2+b^2}{2}}$, numerical mean- $\frac{a+b}{2}$, geometric mean- $\sqrt{ab}$, harmonic mean- $\frac{2ab}{a+b}$ of positive numbers a and b does not go beyond the limits. Indeed, for example , $a = 2$ and $b = 8$, that is, if we consider $[2,8]$,

$$\sqrt{\frac{2^2+8^2}{2}} > \frac{2+8}{2} > \sqrt{2 \cdot 8} > \frac{2 \cdot 8}{2+8}$$

in other words,

$\sqrt{34} > 5 > 4 > 1{,}6$. It's obvious that $\sqrt{34} \in [2,8];\ 5 \in [2,8];\ 4 \in [2,8];\ 1{,}6 \in [2,8]$. Now let's assume that $[a,b] = [t_0, T]$.Then it is obvious that inequality (9) will have the form:

$$\sqrt{\frac{t_0{}^2+T^2}{2}} > \frac{t_0+T}{2} > \sqrt{t_0 T} > \frac{2t_0 T}{t_0+T}, \tag{10}$$

We will use formula (10) as an auxiliary fact for studying the Boltz problem (1)-(3).

III. THE BOLTZ PROBLEM DESCRIBED BY AN INTEGRAL EQUATION AND THE LIPSCHITZ CONDITION IN INTEGRAL FORM

According to the Newton-Leibniz formula, known from integral calculus, the trajectory $x(t),\ t \in [t_0, T]$ can be described as follows:

$$x(t) = x(t_0) + \int_{t_0}^{t} \dot{x}(\tau)d\tau, \tag{11}$$

In fact, formula (11) shows that the trajectory is an absolutely continuous function. An absolutely continuous function, as is known, is a continuous function.

If we consider the right side of equation (1) and the right side of the initial condition (2) in formula (11), we obtain:

$$x(t) = x_0 + \int_{t_0}^{t} F(\tau, x(\tau), u(\tau))d\tau, \tag{12}$$

Thus, equation (1), which describes the controlled process, was reduced to the Volterra integral equation of the second kind in the form (12) with initial conditions (2). Since this integral equation contains both the differential equation (1) and the initial condition (2), then for each fixed control function $u(t) \in U_m$ (12) we will call the integral equation equivalent to the Cauchy problem (1)-(2). In other words, equation (12) is called an equivalent integral equation.

Thus, the Boltz problem, described by the integral equation, can be expressed as follows:

It is required to find the control function $u(t) \in U_m$ corresponding to the solution $x(t)$ of the integral equation (12), which gives a minimum to the functional (3).

Obviously, one of the most frequently used conditions in the mathematical theory of optimal control is the Lipschitz condition. The Lipschitz condition is derived from Lagrange's theorem, which is one of the mean value theorems expressing the constraint on the derivative. As is known from classical mechanics, the derivative of a trajectory with respect to time is speed:

$\dot{x}(t) = \upsilon(t)$. Then it is clear that formula (11) can be written as follows:

$$x(t) = x(t_0) + \int_{t_0}^{t} \upsilon(\tau)d\tau .$$

The physical meaning is that if a material point moves rectilinearly with a constant speed, then the entire trajectory can be determined if the initial position $x(t_0)$ at the beginning of time $t = t_0$ and the velocity of the trajectory are known. In practical optimal control problems, speed is often taken as control: $u(t) = \upsilon(t)$. From a practical point of view, since the control function $u(t)$ is limited, a constraint is imposed on the time derivative of the trajectory, in other words, the Lipschitz condition:

$$|x(t) - x(t_0)| \le L|t - t_0|, \ t \in [t_0, T], \tag{13}$$

Here is the L - Lipschitz coefficient. If we consider formula (12) in condition (13), we obtain:

$$\left| \int_{t_0}^{t} F(\tau, x(\tau), u(\tau)) d\tau \right| \le L|t - t_0|, \ t \in [t_0, T], \tag{14}$$

Note that condition (14) is called the Lipschitz condition in integral form.

IV. New integral inequalities for the trajectory in the study of the Boltz type optimal control problem

In the previous section, we discussed the boundedness of the derivative trajectory and noted the importance of the Lipschitz condition for it. In this section, if the trajectory $x(t), t \in [t_0, T]$ is an strictly monotonically increasing or decreasing function, in other words, if the derivative of the trajectory is positive or negative, then we are talking about obtaining new estimates for it in the form of integral inequalities. Thus, the goal of this section is to obtain new integral inequalities for the trajectory $x(t), t \in [t_0, T]$when it is a monotonic function.

Theorem1. If the trajectory $x(t), t \in [t_0, T]$ in the Boltz problem (1)-(3) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) > x(\frac{t_0+T}{2}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau >$$

$$> x(\sqrt{t_0 T}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau >$$

$$> x(\frac{2t_0 T}{t_0+T}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau \ , \tag{15}$$

Proof. According to the Newton-Leibnitz formula known from integral calculus,

$$x(t) = x(\sqrt{\frac{t_0^2+T^2}{2}}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} \dot{x}(\tau) d\tau, \tag{16}$$

If we take into account that in formula (16) $\dot{x}(t) = F(t, x(t), u(t))$, and also the function $x(t)$ is a strictly monotonically increasing function, and using formula (10) we obtain:

$$x(t) = x(\sqrt{\frac{t_0^2+T^2}{2}}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} \dot{x}(\tau) d\tau =$$

$$= x(\sqrt{\frac{t_0^2+T^2}{2}}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau >$$

$$> x(\frac{t_0+T}{2}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau >$$

$$> x(\sqrt{t_0 T}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau > \tag{17}$$

$$> x(\frac{2t_0 T}{t_0+T}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau$$

As can be seen, the integral inequalities obtained in the formula (17) show the truth of the formula (15) in Theorem 1.

Theorem2. If the trajectory $x(t), t \in [t_0, T]$ in the Boltz problem (1)-(3) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) < x(\frac{t_0+T}{2}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau <$$

$$< x(\sqrt{t_0 T}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau <$$

$$< x(\frac{2t_0 T}{t_0+T}) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau)) d\tau.$$

Note. The proof of Theorem 2 is carried out similarly to Theorem 1.

Theorem3. If the trajectory $x(t), t\in[t_0,T]$ in the Boltz problem (1)-(3) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) > x(\sqrt{t_0T}) + \int_{\frac{t_0+T}{2}}^{t} F(\tau,x(\tau),u(\tau))d\tau >$$
$$> x(\frac{2t_0T}{t_0+T}) + \int_{\frac{t_0+T}{2}}^{t} F(\tau,x(\tau),u(\tau))d\tau, \tag{18}$$

Proof. If the function $x(t)$ is differentiable with respect to time $t\in[t_0,T]$, then it can be described as follows:

$$x(t) = x(\frac{t_0+T}{2}) + \int_{\frac{t_0+T}{2}}^{t} \dot{x}(\tau)d\tau, \tag{19}$$

If we take into account that in formula (19) $\dot{x}(t) = F(t,x(t),u(t))$, and also the function $x(t)$ is a strictly monotonically increasing function, and using formula (10) we obtain:

$$x(t) = x(\frac{t_0+T}{2}) + \int_{\frac{t_0+T}{2}}^{t} \dot{x}(\tau)d\tau =$$
$$= x(\frac{t_0+T}{2}) + \int_{\frac{t_0+T}{2}}^{t} F(\tau,x(\tau),u(\tau))d\tau >$$
$$> x(\sqrt{t_0T}) + \int_{\frac{t_0+T}{2}}^{t} F(\tau,x(\tau),u(\tau))d\tau >$$
$$> x(\frac{2t_0T}{t_0+T}) + \int_{\frac{t_0+T}{2}}^{t} F(\tau,x(\tau),u(\tau))d\tau, \tag{20}$$

As can be seen, the integral inequalities obtained in the formula (20) show the truth of the formula (18) in Theorem 3.

Theorem4. If the trajectory $x(t), t\in[t_0,T]$ in the Boltz problem (1)-(3) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) < x(\sqrt{t_0T}) + \int_{\frac{t_0+T}{2}}^{t} F(\tau,x(\tau),u(\tau))d\tau <$$
$$< x(\frac{2t_0T}{t_0+T}) + \int_{\frac{t_0+T}{2}}^{t} F(\tau,x(\tau),u(\tau))d\tau$$

Note. The proof of Theorem 4 is carried out similarly to Theorem 3.

Theorem5. If the trajectory $x(t), t\in[t_0,T]$ in the Boltz problem (1)-(3) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) > x(\frac{2t_0T}{t_0+T}) + \int_{\sqrt{t_0T}}^{t} F(\tau,x(\tau),u(\tau))d\tau, \tag{21}$$

Proof. Since the function $x(t)$ is differentiable with respect to time $t\in[t_0,T]$, it can be expressed as follows:

$$x(t) = x(\sqrt{t_0T}) + \int_{\sqrt{t_0T}}^{t} \dot{x}(\tau)d\tau, \tag{22}$$

If we take into account that in formula (22) $\dot{x}(t) = F(t,x(t),u(t))$, and also the function $x(t)$ is a strictly monotonically increasing function, and using formula (10) we obtain:

$$x(t) = x(\sqrt{t_0T}) + \int_{\sqrt{t_0T}}^{t} \dot{x}(\tau)d\tau =$$
$$= x(\sqrt{t_0T}) + \int_{\sqrt{t_0T}}^{t} F(\tau,x(\tau),u(\tau))d\tau >$$
$$> x(\frac{2t_0T}{t_0+T}) + \int_{\sqrt{t_0T}}^{t} F(\tau,x(\tau),u(\tau))d\tau, \tag{23}$$

As can be seen, the integral inequalities obtained in the formula (23) show the truth of the formula (21) in Theorem 5.

Theorem6. If the trajectory $x(t), t\in[t_0,T]$ in the Boltz problem (1)-(3) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) < x(\frac{2t_0T}{t_0+T}) + \int_{\sqrt{t_0T}}^{t} F(\tau,x(\tau),u(\tau))d\tau.$$

Note. The proof of Theorem 6 is carried out similarly to Theorem 5.

Theorem7. If the trajectory $x(t), t\in[t_0,T]$ in the Boltz problem (1)-(3) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) > \frac{x(\frac{t_0+T}{2}) + x(\sqrt{t_0T})}{2} + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau,x(\tau),u(\tau))d\tau.$$

Proof. As is known from formula (10),

$$\sqrt{\frac{t_0^{\,2}+T^2}{2}} > \frac{t_0+T}{2},$$

$$\sqrt{\frac{t_0^{\,2}+T^2}{2}} > \sqrt{t_0T}.$$

If we assume that it is a $x(t)$ strictly monotonically increasing function, then it immediately follows that

$$x\left(\sqrt{\frac{t_0^2+T^2}{2}}\right) > x\left(\frac{t_0+T}{2}\right),$$

$$x\left(\sqrt{\frac{t_0^2+T^2}{2}}\right) > x\left(\sqrt{t_0T}\right).$$

If we sum the above inequalities with the same sign side by side and divide each side by 2, we get:

$$x\left(\sqrt{\frac{t_0^2+T^2}{2}}\right) > \frac{x\left(\frac{t_0+T}{2}\right) + x\left(\sqrt{t_0T}\right)}{2}, \qquad (24)$$

If the $x(t)$ function is differentiable in time $[t_0, T]$, then it can be described as follows:

$$x(t) = x\left(\sqrt{\frac{t_0^2+T^2}{2}}\right) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} \dot{x}(\tau)d\tau, \qquad (25)$$

If we consider that in formula (25) $\dot{x}(t) = F(t, x(t), u(t))$ and inequality (24), we get:

$$x(t) = x\left(\sqrt{\frac{t_0^2+T^2}{2}}\right) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} \dot{x}(\tau)d\tau =$$

$$= x\left(\sqrt{\frac{t_0^2+T^2}{2}}\right) + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau))d\tau >$$

$$> \frac{x\left(\frac{t_0+T}{2}\right) + x\left(\sqrt{t_0T}\right)}{2} + \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau))d\tau .$$

The theorem has been proven.

Theorem8. If the trajectory $x(t), t \in [t_0, T]$ in the Boltz problem (1)-(3) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) < \frac{x\left(\frac{t_0+T}{2}\right) + x\left(\sqrt{t_0T}\right)}{2} +$$

$$+ \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau))d\tau.$$

Theorem9. If the trajectory $x(t), t \in [t_0, T]$ in the Boltz problem (1)-(3) is a strictly monotonically increasing function, then the following inequality is true:

$$x(t) > \frac{x\left(\frac{t_0+T}{2}\right) + x\left(\sqrt{t_0T}\right) + x\left(\frac{2t_0T}{t_0+T}\right)}{3} +$$

$$+ \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau))d\tau.$$

Theorem10. If the trajectory $x(t), t \in [t_0, T]$ in the Boltz problem (1)-(3) is a strictly monotonically decreasing function, then the following inequality is true:

$$x(t) < \frac{x\left(\frac{t_0+T}{2}\right) + x\left(\sqrt{t_0T}\right) + x\left(\frac{2t_0T}{t_0+T}\right)}{3} +$$

$$+ \int_{\sqrt{\frac{t_0^2+T^2}{2}}}^{t} F(\tau, x(\tau), u(\tau))d\tau.$$

Note. The proof of Theorem 8, Theorem 9 and Theorem 10 is carried out similarly to Theorem 7.

CONCLUSION

The paper demonstrates Boltz type optimal control problems described by integral inequalities.

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