On Some Ways to Construct Multistep Methods with High Degree and Their Application to Solving Initial-Value Problem for Odes of the Second Order

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***Abstract*— Among the numerical methods constructed for solving initial-value problems for Ordinary Differential Equations (ODEs), the most popular are the Runge-Kutta and Adams methods. Each of these methods has its advantages and disadvantages. For comparison of Runge-Kutta and Adams methods with the multistep methods have used the conception of stability, degree, region of stability, and volume of computational works. As is known many scientists have investigated the numerical solution of the initial-value problem of ODEs of the first and second order, since the Age of Newton. Some specialists tried to apply some modification of methods, which are constructed for solving initial-value problems for the ODEs of the first and second order. Considering the actuality of the above-suggested numerical methods, here have compared the known classes' methods and constructed the new methods on the intersection of the above-mentioned methods. To use above mentioned, here have constructed Multistep second and third derivatives Methods and proven the advantages of the suggested methods.**

Keywords— Initial-value problem, Stability and Degree, Multistep second and third derivative methods.

I. INTRODUCTION

Störmer in solving problem on the dynamics of a charged particle in super position of Dipole and homogeneous magnetic fields, received some Ordinary Differential Equations for solving which can not been fined any Numerical Methods for solving above named problem, which in more general form has written in the following form (see for example [1]-[26]):

$$y''(x) = f(y,x),\ \ y(x_0) = y_0,\ y'(x_0) = y'_0,\ x_0 \le x \le X. \quad (1)$$

For the sake of objectivity, let us noted that, at that time were known only the methods of the Adams and Runge-Kutta. Störmer by using the following known equality:

$$y_{n+1} - 2y_n + y_{n-1} = h^2 y''_n + O(h^4),$$

has constructed of his known method, which can be presented as:

$$y_{n+k} = 2y_{n+k-1} - y_{n+k-2} + h^2 \sum_{i=0}^{k} \beta_i y''_{n+i}. \quad (2)$$

Here suppose that the problem (1) has the continuous unique solution $y(x)$, which in the segment $[x_0, X]$, where has the continuous derivatives up to p+2, inclusively. But the continuous to totality of arguments function $f(x,y)$ has defined in some closed area in which has the partial derivatives up to some p, inclusively. In the method (2) by the y_n has defined the approximately values of the solution of problem (1), at the point x_n. But the corresponding exact values of $y(x)$ have defined by the $y(x_n)$. By the $x_{i+1} = x_i + h\,(i = 0,1,..,N-1)$ have defined the mesh-pointers and by the $h > 0$ the step-size which is divided the segment $[x_0, X]$ to N equal parts. Note that in the method (2), depends from the value of β_k, method is called in different form. If $\beta_k = 0$, then method (2) called as the explicit and in this case the coefficients $\beta_0, \beta_1,.., \beta_{k-1}$ can been defined by some definite integral. As is known, the classic form of presentation of the explicit methods of Störmer type can be written as the follows:

$$y_{n+1} = 2y_n - y_{n-1} = h^2 \sum_{i=0}^{k} \overline{\beta}_i \nabla^i f_n, \quad (3)$$

here $\nabla^i f_n = \nabla(\nabla^{i-1} f_n) = \nabla^{i-1} f_n - \nabla^{i-1} f_{n-1}$,

$$\overline{\beta}_i = \frac{1}{i!}\left(\int_0^1 (1-t)t(t+1)...(t+(i-1))dt - \int_0^{-1} (1+t)t..(t+(i-1))dt\right), \quad i = 0,1,...,k-1.$$

this defined integral can be written as follows:

$$\beta_i = \frac{1}{i!}\int_{-1}^{1} t(t+1)....(t+i-1)\big((1-|t|\big)dt,\ \ i = 0,1,2,..,k-1.$$

Let us method (3) to write as the following:

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=0}^{k} \gamma_i f_{n+i},\ n = 0,1,..,N-k, \quad (4)$$

here

$$\gamma_i = (-1)^i \left\{ \binom{i}{i}\overline{\beta}_i + \binom{i+1}{i}\overline{\beta}_{i+1} + ... + \binom{k}{i}\overline{\beta}_k \right\},\ (i = 0,1,..,k).$$

Noted that the methods (4) and (2) has some similarities, but they are note the same. For each fixed value of k, in the class of methods (4) there is only one method, the accuracy for which is fixed. Therefore, for each value of k, there is only one method, which has a certain accuracy. As is follows from the above descript, the calculations of the coefficients γ_i to respect above given scheme is difficult. Note that in solving some problems it becomes necessary to construct a method with some additional property, for example extended stability region. Taking into account such cases, scientists conclude that in order to fine the coefficients in the method (2) it is necessary to use a different scheme. Some authors for this aim have suggested using method of unknown is coefficients. For the illustration of this Dahlquist to consider more general case than the method (2), having the following form (see for example [21]-[37]):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h^2\sum_{i=0}^{k}\gamma_i f_{n+i},\ n=0,1,2,..,N-k. \quad (5)$$

The aim of this investigation of the construction the stable Multistep Method with high order, which is related to choosing the coefficients.

For the finding, the values of the coefficients of method (5), the scientists have suggested to use solution of the following system of linear algebraic equation:

$$\sum_{i=0}^{k}\alpha_i = 0;\ \sum_{i=0}^{k} i\alpha_i = 0;\ \sum_{i=0}^{k}\gamma_i = \sum_{i=0}^{k}\frac{i^2}{2!}\alpha_i;...; \\ \sum_{i=0}^{k}\frac{i^{p-2}}{(p-2)!}\gamma_i = \sum_{i=0}^{k}\frac{i^p}{p!}\alpha_i. \quad (6)$$

As is known for comparison methods of type (5) in basically are used the conception of stability and degree, which can be presented as follows (see for example [38]-[61]):

Definition 1. The integer value p, is called as the degree for the method (5) if the following asymptotic equality

$$\sum_{i=0}^{k}(\alpha_i y(x+ih) - h^2\gamma_i y''(x+ih)) = O(h^{p+2}),\ h\to 0, \quad (7)$$

is hold.

Definition 2. Method of (5) is called as the stable if the roots of the polynomial

$$\rho(\lambda) \equiv \alpha_k\lambda^k + \alpha_{k-1}\lambda^{k-1} + ... + \alpha_1\lambda + \alpha_0$$

located in the unite circle on the boundary of which there is not multiple roots.

It is easy to understand that, the method (5) can be generalized in different form. For example:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y''_{n+i} + h^3\sum_{i=0}^{k}\gamma_i y'''_{n+i},\ n=0,1,..,N-k. \quad (8)$$

Or in the following form:

$$\sum_{i=0}^{k}\bar{\alpha}_i y_{n+i} = h\sum_{i=0}^{k}\bar{\beta}_i y'_{n+i} + h^2\sum_{i=0}^{k}\bar{\gamma}_i y''_{n+i},\ n=0,1,..,N-k. \quad (9)$$

Note that other cases are also possible.

The stability of the method (8), one can be defined by using definition 2, but the conception of stability of the method (9) can be defined as the following: $|\bar{\beta}_0| + |\bar{\beta}_1| + .. + |\bar{\beta}_k| \neq 0.$

Definition 3. The method (9) is called as the stable if the roots of the polynomial

$$\bar{\rho}(\lambda) = \bar{\alpha}_k\lambda^k + \bar{\alpha}_{k-1}\lambda^{k-1} + ... + \bar{\alpha}_1\lambda + \bar{\alpha}_0$$

are located in the unite circle, on the boundary of which there is not multiply root.

If in the method (9) to consider the case $\bar{\beta}_i = 0\,(i=0,1,..,k),$ then from the method (9) it follows the method (5). Noted that the conception of degree in case $|\bar{\beta}_0| + |\bar{\beta}_1| + .. + |\bar{\beta}_k| \neq 0$ for the method (9) can been defined as follows.

Definition 4. The integer value p is called as the degree for the method (9), if the following is holds:

$$\sum_{i=0}^{k}(\bar{\alpha}_i y(x+ih) - h\bar{\beta}_i y'(x+ih) - h^2\bar{\gamma}_i y''(x+ih)) = O(h^{p+1}),\ h\to 0.$$

By simple comparison, receive that the properties of the stability of above considered Multistep Methods depends on the values of the coefficients $\bar{\beta}_i\,(i=0,1,2,..,k)$. Noted that the stability properties of Multistep Methods depend on the values of the coefficients $\alpha_i\ (i=0,1,..,k)$. But the value of the degree of these methods depends on the values of other coefficients, that is selection of coefficients is the main in investigation of the Multistep Methods. The specified property is true if the following conditions are satisfies:

$$\sum_{i=0}^{k}|\beta_i| \neq 0;\ \sum_{i=0}^{k}|\gamma_i| \neq 0;\ \sum_{i=0}^{k}|l_i| \neq 0;$$

for the following method:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i} + h^2\sum_{i=0}^{k}\gamma_i y''_{n+i} + h^3\sum_{i=0}^{k}l_i y'''_{n+i}. \quad (10)$$

It is easy to determine that the study of method (10) is more difficult than the previous similar methods. For example the cases:

I. $$\sum_{i=0}^{k}|\beta_i| \neq 0;\ \sum_{i=0}^{k}|\gamma_i| = 0;\ \sum_{i=0}^{k}|l_i| \neq 0,$$

II. $$\sum_{i=0}^{k}|\beta_i| = 0;\ \sum_{i=0}^{k}|\gamma_i| = 0;\ \sum_{i=0}^{k}|l_i| \neq 0.$$

Method (8) is also partial case of the method (10). Consequently investigation method (10) is more complex than the previous ones. Let us consider the following method:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h^3\sum_{i=0}^{k}l_i y'''_{n+i},\ n=0,1,2,..,N-k, \quad (11)$$

which is the partial case of method (10).

By using above described method, one can be prove that for the convergence of method (11) the $\lambda = 1$ must be triple root.

By using above described schemes, one can be gave the following definition for the stability of the method (11).

Definition 5. Method (11) is called as the stable, if the roots of the polynomial $\rho(\lambda)$ located in the unite circle on the boundary of which there is not multiple root, except for the triple root $\lambda = 1$.

But the conception of degree for the method (11) can be formulated as following:

Definition 6. Integer value p is called as the degree for the method (11) if the following is holds:

$$\sum_{i=0}^{k}(\alpha_i y(x+ih)-h^3 l_i y'''(x+ih))=O(h^{p+3}),\ h\to 0. \quad (12)$$

The relationship between of the order k and the degree p of the method (11) is as follows:

$$p\le 2[k/2]+2$$

in the case $l_k \neq 0$ and $p\le k$ in the case $l_k = 0$. For each k -there is the stable method with maximum degree.

It is not difficult determine the maximum value for the degree of Störmer method. By taking into account that the Störmer method is stable, then receive that the degree of the Störmer method will by satisfy the following condition: $p\le 2[k/2]+2$ for the case $\beta_k \neq 0$ and $p\le k$ for the $\beta_k = 0$. For the construction more exact Multistep methods, let us consider method (9). For the estimation the exactness of the method (9), let us consider the following theorem:

Theorem1: If method (9) is stable and has the degree of p, then $p\le 2k+2$ for the implicit methods and $p\le 2k$ for explicit methods. There exist the stable methods with the maximum degree for each k. If method (9) is instable, then there are methods of type (9) with degree $p\le 3k+1$.

One of the classic Störmer methods can be presented as follows:

$$y_{n+2}=2y_{n+1}-y_n+h^2(y''_{n+2}+10y''_{n+1}+y''_n)/12. \quad (13)$$

This method is stable and has the degree p=4. This method is very often in solving practical problems. For the illustration of the advantages of Störmer methods, in the following section study the application of Störmer method to solve the model problem.

II. Numerical Results

Here, consider application method (13) to solve the following model problem:

$$y''=\lambda^2 y(x),\ \ y(0)=1,\ y'(0)=\lambda,\ 0\le x\le 1. \quad (14)$$

Method (13) is implicit, therefor in the results of application of method (13) to solve some problems we are faced with solving nonlinear algebraic equations.

As is known in this case, usually specialists are used the predictor-corrector methods. In our case as the predictor method one can used the following:

$$\hat{y}_{n+2}=2y_{n+1}-y_n+h^2 f_{n+1}.$$

In this case the Störmer method can be written as follows:

$$y_{n+2}=2y_{n+1}-y_n+h^2(\hat{y}''_{n+2}+10y''_{n+1}+y''_n)/12. \quad (15)$$

The results receiving in the application method (15) to solve the problem (14) are tabulated in the table 1.

Table 1. Error for method (15) at $\lambda=1$

Variable x	Step Size $h=0.1$	Step Size $h=0.05$	Step Size $h=0.01$
0.20	4.5E-09	3.8E-10	7.32E-13
0.60	5.29E-08	3.45E-09	5.68E-12
1.00	1.2E-07	7.51E-09	1.2E-11

Below have presented some methods that are good illustrate above receiving results:

$$y_k=y_{k-2}+2h(2f_{k-1}+8f_{k-1}+5f_{k-2})/15$$
$$+2h^2(2g_{k-2}+g_{k-3})/5,\ R_n=-h^6 y^{(6)}/900, p=5. \quad (16)$$

$$y_k=y_{k-1}+h(101f_k+128f_{k-1}+11f_{k-2})/240$$
$$+h^2(-13g_k+40g_{k-1}+3g_{k-2})/240;\ R_n=o(h^7),\ p=6. \quad (17)$$

$$y_k=(13y_{k-1}-y_{k-2})/12+h(-8939f_{k+1}+184599f_k$$
$$+168507f_{k-1}-11527f_{k-2})/362880 \quad (18)$$
$$+h^2(1241g_{k+1}-7389g_k+12753g_{k-1}-501g_{k-2})/362880;$$
$$R_n=10^{-5}h^9 y^9, p=8.$$

III. Conclusion

In solving many applied problems, more exact numerical methods a needed. As was shown here in the application of the methods with high degree, some difficulties usually arise in the application of these methods. In the noted difficulties associated with that, these methods usually are implicit, there for it is necessary to use them to use the predictor-corrector methods. As shown above, the methods of calculating the higher derivatives of the solution of the investigated problem are more accurate. For example method (10) is more exact than the methods (5), (8), and (9). For the sake of objectivity, note that each method has its advantages and disadvantages. Their specialists suggest using simple methods with a high degree, which is usually called the optimal methods.

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Conflict to Interest

The authors state express that there is no conflict of interest misunderstanding between them.

We hereby confirm that all the methods in this manuscript are ours.

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