An Optimal Control Problem for an Elliptic Equation

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***Abstract*—An optimal control problem for an elliptic equation with periodic boundary condition is considered. The correctness of the problem statement is investigated. The existence and uniqueness of the generalized solution to the boundary problem for the fixed controls is proved and a priori estimation for its solution is obtained. The theorem on the existence and uniqueness of the solution to the considered optimal control problem is proved. It is proved that the objective functional is differentiable on the space of controls, and an explicit expression for its gradient is derived. Necessary and sufficient conditions for the optimality of the controls are obtained.**

***Keywords*—elliptic equation, optimal control, periodicity conditions, correctness of problem statement, optimality criterion.**

I. Introduction

Optimal control problems for elliptic equations arise in the fields of elasticity theory, heat transfer, convection-diffusion-reaction, and environmental forecasting [1-3]. Such problems have been studied quite thoroughly for elliptic equations with classical boundary conditions [4-7]. However, these problems have been less studied in the case of periodic boundary conditions.

II. Statement of the Problem and and Its Correctness

Let $\Omega = \{x = (x_1, \dots, x_2) \in R^n : 0 < x_i < l_i,\ i = \overline{1,n}.\}$ be a parallelepiped in R^n. Let us denote by $\widehat{W}_2^1(\Omega)$ the subspace formed by the l-periodic elements of $W_2^1(\Omega)$, i.e. the elements $u = u(x) \in W_2^1(\Omega)$ satisfying the condition $u|_{x_i=0} = u|_{x_i=l_i}, i = \overline{1,n}$. In this work, the scalar product in the space $L_2(\Omega)$ is denoted by (u, v), and the norm is denoted by the symbol $\|u\|$. Suppose that the controlled process is described as follows:

$$Lu \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial u}{\partial x_i}\right) + a(x)u = v(x), \quad x \in \Omega, \quad (1)$$

$$u|_{x_i=0} = u|_{x_i=l_i}, \qquad i = \overline{1,n}, \quad (2)$$

$$k(x)\frac{\partial u}{\partial x_i}|_{x_i=0} = k(x)\frac{\partial u}{\partial x_i}|_{x_i=l_i}, \qquad i = \overline{1,n}. \quad (3)$$

Here the functions $k(x), a(x)$ are given, $v = v(x)$ is a control function that satisfies the following condition:

$$v = v(x) \in V \subset L_2(\Omega). \quad (4)$$

Here $\alpha \geq 0$ is a given number, $z(x) \in L_2(\Omega)$ is a given function.

Let us consider the following optimal control problem:

The problem (1)-(3) consists in minimizing the functional

$$\mathcal{J}(v) = \int_\Omega |u(x,v) - z(x)|^2 dx + \alpha \int_\Omega v^2(x)\, dx \quad (5)$$

on the set of solutions $u = u(x) = u(x, v)$ corresponding to the control $v \in L_2(\Omega)$. Here the functions $k(x), a(x)$ are measurable functions satisfying the following conditions:

$$0 \leq \nu \leq k(x) \leq \mu, 0 < \mu_1 \leq a(x) \leq \mu_2, \qquad x \in \Omega, \quad (6)$$

$$\nu, \mu, \mu_1, \mu_2 = const > 0.$$

Definition 1. The generalized solution to the boundary value problem (1)-(3) in the space $W_2^1(\Omega)$ is called the function $u = u(x) = u(x,v) \in \widehat{W}_2^1(\Omega)$ such that it satisfies the following integral identity for $\forall \eta \in \widehat{W}_2^1(\Omega)$:

$$L(u,\eta) \equiv \int_\Omega \left(\sum_{i=1}^{n} k(x)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_i} + a(x)u\eta\right) dx =$$

$$= \int_\Omega v(x)\eta dx. \quad (7)$$

Theorem 1. Assume that conditions (6) are satisfied. Then, for each fixed $v = v(x) \in L_2(\Omega)$, there can be at most one generalized solution to the boundary value problem (1)-(3) in the space $W_2^1(\Omega)$ and the estimate

$$\|u\|_{W_2^1(\Omega)} \leq c_1\|v\| \quad (8)$$

is hold. Here $c_1 > 0$ is a constant.

Proof. Consider the quadratic form $L(u, u)$. If we choose $\eta = u$ in (7) and taking into account the inequalities (6), we obtain

$$L(u,u) = \int_\Omega \left[\sum_{i=1}^{n} k(x)\left(\frac{\partial u}{\partial x_i}\right)^2 + a(x)u^2\right] dx \geq$$

$$\geq \int_\Omega (\nu u_x^2 + \mu_1 u^2) dx = \nu\|u_x\|^2 + \mu_1\|u\|^2. \quad (9)$$

According to "Cauchy inequality with ε" when $\varepsilon = \frac{\mu_1}{2}$, we obtain the following inequality:

$$L(u,u) = (v,u) \leq \|v\|\|u\| \leq \frac{\mu_1}{2}\|u\|^2 + \frac{1}{2\mu_1}\|v\|^2 \quad (10)$$

It follows from (9) and (10) that:

$$\nu\|u_x\|^2 + \frac{\mu_1}{2}\|u\|^2 \leq \frac{1}{2\mu_1}\|v\|^2.$$

Hence we obtain the following two inequalities:

$$\|u_x\|^2 \le \frac{1}{2\nu\mu_1}\|v\|^2,$$

$$\|u\|^2 \le \frac{1}{{\mu_1}^2}\|v\|^2.$$

If we add these inequalities side by side and take the square root of the obtained inequality, we obtain estimation (8) with the constant $c_1 = \left(\frac{1}{2\nu\mu_1} + \frac{1}{{\mu_1}^2}\right)^{\frac{1}{2}}$ .The uniqueness of the solution to the problem (1)-(3) is obtained from the estimation (8). The theorem is proved.

Theorem 2. If conditions (6) are satisfied, then for every $v \in L_2(\Omega)$ there exists a generalized solution to the problem (1)-(5) from the space $W_2^1(\Omega)$.

Proof. Let us define a new scalar product in the space $W_2^1(\Omega)$:

$$[u,v] = \int_\Omega \left(\sum_{i=1}^n k(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + a(x)uv\right)dx.$$

This scalar product is identical to the following scalar product of the space $W_2^1(\Omega)$:

$$(u,v) = \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + uv\right)dx.$$

Therefore, the identity (7) can be represented as follows:

$$[u,\eta] = (v,\eta) \qquad (11)$$

When $v \in L_2(\Omega)$ is specified, the expression (v,η) defines a linear functional depending on η in the space $\widehat{W}_2^1(\Omega)$. Besides that, since

$$|(v,\eta)| \le \|v\|\|\eta\| \le c_2\|v\|\|\eta\|_{\widehat{W}_2^1(\Omega)}$$

the functional is bounded. Here the constant $c_2 > 0$ does not depend on v and η. Then according to Ritz theorem, there is a unique function $F \in \widehat{W}_2^1(\Omega)$ such that $(v,\eta) = [F,\eta]$, $\forall\eta \in \widehat{W}_2^1(\Omega)$. Thus, there is only one function $u = F$ in space $W_2^1(\Omega)$ which satisfies the identity (8). The theorem is proved.

Theorem 3. Suppose that conditions (6) are satisfied and the set V is a closed, convex and bounded set in the space $L_2(\Omega)$. Then the optimal controls set V_* of problem (1)-(5) is not empty, but is closed, convex and bounded, and any minimizing sequence in the space $L_2(\Omega)$ converges weakly to the set V_*. Moreover, if $\alpha > 0$, then the set V_* consists in the only one point $v_* = v_*(x) \in V$, and the arbitrary minimizing sequence converges to the element v_* according to the norm of the space $L_2(\Omega)$.

III. Differention of the Objective Functional and Optimality Criterion

Consider the adjoint problem corresponding to the problem (1)-(3):

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial\psi}{\partial x_i}\right) + a(x)\psi =$$

$$= 2[u(x) - z(x)], \qquad x \in \Omega, \qquad (12)$$

$$\psi|_{x_i=0} = \psi|_{x_i=l_i}, \qquad i = \overline{1,n}, \qquad (13)$$

$$k(x)\frac{\partial\psi}{\partial x_i}|_{x_i=0} = k(x)\frac{\partial\psi}{\partial x_i}|_{x_i=l_i}, \qquad i = \overline{1,n}. \qquad (14)$$

Definition 2. The generalized solution to the boundary value problem (12)-(14) in the space $W_2^1(\Omega)$ is called the function $\psi = \psi(x) = \psi(x,v) \in \widehat{W}_2^1(\Omega)$ such that it satisfies the following integral identity for $\forall\eta \in \widehat{W}_2^1(\Omega)$:

$$\int_\Omega \left(\sum_{i=1}^n k(x)\frac{\partial\psi}{\partial x_i}\frac{\partial\eta}{\partial x_i} + a\psi\eta\right)dx =$$

$$= 2\int_\Omega [u(x,v) - z(x)]\eta dx. \qquad (15)$$

From Theorems 1 and 2, it follows that for each $v \in L_2(\Omega)$ there exists a unique generalized solution to the problem (12)-(14) from the space $W_2^1(\Omega)$ and the following estimation is hold:

$$\|\psi\|_{W_2^1(\Omega)} \le c_3\|[u(x,v) - z(x)]\|.$$

Here $c_3 > 0$ is a constant.

Let us show that the functional (5) is differentiable in the space $L_2(\Omega)$ under conditions (1)-(3). For this purpose, let us choose the controls $v, v + \Delta v \in L_2(\Omega)$ and assume that $\Delta u(x,v) = u(x, v + \Delta v) - u(x,v)$. Then from the conditions (1)-(3), it follows that Δu is the generalized solution to the following boundary problem:

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial\Delta u}{\partial x_i}\right) + a\Delta u = \Delta v, \qquad x \in \Omega, \qquad (16)$$

$$\Delta u|_{x_i=0} = \Delta u|_{x_i=l_i}, \qquad i = \overline{1,n}, \qquad (17)$$

$$k(x)\frac{\partial\Delta u}{\partial x_i}|_{x_i=0} = k(x)\frac{\partial\Delta u}{\partial x_i}|_{x_i=l_i}, \qquad i = \overline{1,n}. \qquad (18)$$

A generalized solution to this problem satisfies the following integral identity for arbitrary $\eta = \eta(x) \in \widehat{W}_2^1(\Omega)$:

$$\int_\Omega \left(\sum_{i=1}^n k(x)\frac{\partial\Delta u}{\partial x_i}\frac{\partial\eta}{\partial x_i} + a\Delta u\eta\right)dx = \int_\Omega \Delta v\eta dx, \qquad (19)$$

The estimation (8) in Theorem 1 shows that

$$\|\Delta u\|_{W_2^1(\Omega)} \le c_4\|\Delta v\| \qquad (20)$$

for a function Δu satisfying the identity (19). Here $c_3 > 0$ is a constant which does not depend on Δv.

Now let us consider the increment of the functional (5):

$$\Delta\mathcal{J}(v) = \mathcal{J}(v + \Delta v) - \mathcal{J}(v) =$$

$$= \int_\Omega |u(x, v + \Delta v) - z(x)|^2 dx - \int_\Omega |u(x,v) - z(x)|^2 dx +$$

$$+\alpha\int_\Omega (v + \Delta v)^2 dx - \alpha\int_\Omega v^2 dx =$$

$$= 2\int_{\Omega}[u(x,v) - z(x)]\Delta u(x,v)dx + \int_{\Omega}|\Delta u(x,v)|^2 dx +$$

$$+\alpha\int_{\Omega}(2v\Delta v + \Delta v^2)dx. \qquad (21)$$

If we choose $\eta = \Delta u$ in the identity (15) and $\eta = \psi$ in the identity (19) and subtract the resulting equations side by side, we obtain the following equation

$$2\int_{\Omega}[u(x,v) - z(x)]\Delta u dx = \int_{\Omega}\psi\Delta v dx.$$

Substituting this equation into (20), we obtain

$$\Delta\mathcal{J}(v) = \int_{\Omega}(\psi + 2\alpha v)\Delta v dx + R, \qquad (22)$$

where

$$R = \int_{\Omega}|\Delta u(x,v)|^2 dx + \alpha\int_{\Omega}\Delta v^2 dx.$$

From the estimation (20) and the expression of R, we obtain that $R = o(\|\Delta v\|)$. It follows from (22) that the functional $\mathcal{J}(v)$ is differentiable in the space $L_2(\Omega)$ and its gradient at the point $v \in L_2(\Omega)$ is determined by the equality

$$\mathcal{J}'(v) = \psi + 2\alpha v \in L_2(\Omega). \qquad (23)$$

Thus the following theorem has been proved:

Theorem 4. When conditions (1)-(3) are satisfied the functional (5) is Fréchet differentiable in the space $L_2(\Omega)$ and its gradient is determined by the equality (23).

The following theorem expresses the optimality criterion for the problem (1)-(5).

Theorem 5. The satisfaction of the following inequality is necessary and sufficient for the optimality of control $v_* = v_*(x) \in V$ in problem (1)-(5):

$$\int_{\Omega}(\psi_* + 2\alpha v_*)(v - v_*)dx \geq 0, \qquad (24)$$

$$\forall v = v(x) \in V.$$

Here, the function $\psi_* = \psi(x, v_*)$ is the solution to the problem (12)-(14) corresponding to the control $v = v_*$. If $v_* \in intV$, then the inequality (24) is equivalent to the equality

$$\psi_*(x) + 2\alpha v_*(x) = 0, \qquad x \in \Omega.$$

IV. Conclusion

The correctness of the statement of the optimal control problem for the elliptic equation with periodic boundary conditions is investigated. The differentiability of the objective functional is proven, an explicit expression for its gradient is derived, and the optimality criterion is obtained.

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