Solution of the Boundary Value Problem for Operator-Differential Equations in Hilbert Spaces

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Abstract— In the paper there are Fredholm solvability of a boundary value problem in a finite domain of a class of second-order differential equation of elliptic type in a seperable Hilbert space. Sufficient conditions that provide regular and fredholm solvability of the given problem, are found. These conditions are expressed only by the coefficients.

Keywords— *Hilbert space, linear operator, self adjoint operator, regular solvability,izomorfism, Fredholm solvability.*

I. Introduction

As is known the Cauchy problem was first studied by E.Hille, K. Iosido, T.Kato and others. Further, boundary value problems and related problems have been studied by many authors. Some of these results have found their reflection in the books of A.A. Dezin [6], V.I. Gorbachuk and M.L. Gorbachuk [11], S.Ya. Yakubov [20] and others In an finite domain, boundary value problems with variable coefficients have been studied very little. We can note the works of S.S. Mirzoev with G.A.Agayeva [13,14], G.A.Agayeva [1,2,3].

Let H be a separable Hilbert space. Assume that C is a self-adjoint operator with domain of definition $D(C)$. Then for all $\gamma \geq 0$ the domain of definition of the operator C^{γ} will be a Hilbert space $H_{\gamma}(\gamma \geq 0)$ with a scalar product $(x,y)_{\gamma} = (C^{\gamma}x, C^{\gamma}y)$. For $\gamma = 0$ we assume $H_0 = H$ and $(x,y)_0 = (x,y)$.

Denote by $L_2((0,1):H)$ a Hilbert space of vector –functions determined almost everywhere in $(0,1)$ for which

$$\|f\|_{L_2((0,1):H)} = \left(\int_0^1 \|f(t)\|^2\, dt\right)^{1/2}$$

We determine the Hilbert space

$W_2^2((0,1):H) = \{u : u'' \in L_2((0,1):H),\ C^2 u \in L_2((0,1):H)\}$

with the norm

$$\|u\|_{W_2^2((0,1):H)} = \left(\|u''\|^2_{L_2((0,1):H)} + \|C^2 u\|^2_{L_2((0,1):H)}\right)^{1/2}.$$

We determine the subspace $W_{2,U}^2((0,1):H)$ as follows

$$W_{2,U}^2((0,1):H) = \{u : u \in W_2^2((0,1):H),$$
$$u(0) = \alpha u(1), u'(0) = \beta u'(1), U \in R = (-\infty,\infty)\}$$

Note that for $\alpha = \beta = 1$ we obtain a subspace of periodic functions, for $\alpha = \beta = -1$ we obtain a space of anti periodic functions.

Consider in H the boundary value problem

$$L(d/dt)u(t) = -u''(t) + \rho(t)A^2u(t) + \\ + (A_1 + K_1)u'(t) + (A_2 + K_2)u(t) = f(t),\ t \in (0,1) \quad (1)$$

$$u(0) = \alpha u(1),\ u'(0) = \beta u'(1) \quad (2)$$

where the operator coefficients of the equation (1) satisfy the conditions:

1) A is a normal operator with a completely continuous operator in H, whose set of spectra is contained in the angular sector
$$S_{\varepsilon} = \{\lambda : |\arg\lambda < \varepsilon,\ 0 \leq \varepsilon \leq \pi/4\};$$
2) $\rho(t)$ is a scalar function defined in (0,1), measurable and bounded, moreover $0 < \gamma \leq \rho(t) \leq \delta < \infty$, where $\gamma, \delta \in R = (-\infty,\infty)$;
3) The operators $B_1 = A_1A^{-1}$ *and* $B_2 = A_2A^{-2}$ are bounded in H;
4)The operators $T_1 = K_1A^{-1}$ *and* $T_2 = K_2A^{-2}$ are completely continuous operators in H;
5) $\alpha\beta \neq -1$

Note that subject to the conditions 1), the operator A has an orthonormal basis system in H, i.e. $Ae_k = \lambda_k e_k \quad (k = 1,2,\ldots) \quad 0 < |\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_k| < \ldots$

moreover $(e_k, e_j) = \delta_{k,j} = \begin{cases} 0, k \neq j \\ 1,\ k = j \end{cases}$ and

$\lambda_k = |\lambda_k| e^{i\varphi_k}, \varphi_k \in S_\varepsilon, k = 1,2,...$

$$A(\cdot) = \sum_{k=1}^{\infty} \lambda_k (\cdot, e_k) e_k,$$

$$C(\cdot) = \sum_{k=1}^{\infty} |\lambda_k| (\cdot, e_k) e_k,$$

$$U(\cdot) = \sum_{k=1}^{\infty} e^{i\varphi_k} (\cdot, e_k) e_k, \qquad \varphi_k \in S_\varepsilon, k = 1,2,...$$

In what follows , we will use the theorems on intermediate derivatives and from the trace theorem [12 , p. 31, p. 41], i.e.

1) if $u \in W_2^2((0,1):H)$, then $Cu' \in L_2((0,1):H)$ and

$$\| Cu' \| \le const \| u \|_{W_2^2((0,1):H)},$$

2) if $u(t) \in W_2^2((0,1):H)$, then for any $t_0 \in [0,1]$ $u(t_0)$ and $u'(t_0)$ these exists $u(t_0) \in H_{3/2}$, $u'(t_0) \in H_{1/2}$ and we have the inequality

$$\| u(t_0) \|_{3/2} \le const \| u \|_{W_2^2((0,1):H)} \qquad 0 \le t_0 \le 1$$

and

$$\| u'(t_0) \|_{1/2} \le const \| u \|_{W_2^2((0,1):H)}$$

Definition1. If for $f(t) \in L_2((0,1):H)$ there exist $u(t) \in W_2^2((0,1):H)$, then $u(t)$ is called a regular solution of the equation (1)

Definition2. If for any $f(t) \in L_2((0,1):H)$ there exists a regular solution of equation (1) that satisfies the boundary conditions (2) in the sense of convergence

$$\lim_{t \to 0} \| u(t) - \alpha u(1-t) \|_{3/2} = 0 \ ,$$

$$\lim_{t \to +0} \| u'(t) - \beta u'(1-t) \|_{1/2} = 0 \ ,$$

and we have the estimates

$$\| u(t) \|_{W_2^2((0,1):H)} \le const \| f \|_{L_2((0,1):H)}$$

then problem (1) –(2) is called regularly solvable.

Denote by

$$Lu = \mathrm{P}u + Tu, \quad u \in W_{2,U}^2((0,1):H) \ ,$$

where

$$\mathrm{P}u = -u'' + \rho(t) A^2 u + A_1 u' + A_2 u, \quad u \in W_{2,U}^2((0,1):H)$$

and

$$Ku = K_1 u' + K_2 u, \quad u \in W_{2,U}^2((0,1):H)$$

Definition3. If the operator L mapping $u \in W_{2,U}^2((0,1):H)$ into $L_2((0,1):H)$ is Fredholm, we say that problem (1.1),(1.2) is Fredholm solvable.

II. Some results.

Thus, the set of spectra of the operator A is contained in the spectrum S_ε, i.e. there exists bounded spectra $e^{-At} (t \ge 0)$ generated by the operator A

There are:

Lemma 1. For $\varphi \in H_{3/2}$ the inequality

$$\| e^{-tA} \varphi \|_{W_2^2((0,1):H)} \le \frac{1}{\sqrt{\cos \varepsilon}} \| \varphi \|_{3/2}$$

Lemma 2. Let $x \in D(A)$, then

$$\mathrm{Re}(A^* x, Ax) \ge \cos 2\varepsilon \| Cx \|^2$$

Theorem1. Let conditions 1) and 2) be fulfilled. Then for any $u \in W_{2,U}^2((0,1):H)$ we have the inequality

$$\| Au' \|_{L_2((0,1):H)} \le d_1(\varepsilon) \| P_0 u \|_{L_2((0,1):H)}$$

and

$$\| A^2 u \|_{L_2((0,1):H)} \le d_0(\varepsilon) \| P_0 u \|_{L_2((0,1):H)} \ ,$$

where

$$d_1(\varepsilon) = \frac{1}{2\sqrt{\alpha}} \frac{1}{\cos \varepsilon} \quad , \quad d_0 = \frac{1}{\alpha}.$$

Corollary1. For $u \in W_{2,U}^2((0,1):H)$ we have the inequalities

$$\| A^* u' \|_{L_2((0,1):H)} \le d_1(\varepsilon) \| L^* u \|_{L_2((0,1):H)}$$

and

$$\| A^{*2} u \|_{L_2((0,1):H)} \le d_0(\varepsilon) \| L^* u \|_{L_2((0,1):H)}$$

III. Main results

Here we show the conditions for regular and Fredholm solvability or problem (1),(2)

Theorem 2. The operator L_0 isomorphically maps the space $W_{2,\psi}^2((0,1):H)$ onto the space $L_2((0,1):H$

Theorem 3. Let conditions 1)-4) be fulfilled and we have the inequality

$$q = \frac{1}{2\sqrt{\gamma}} \frac{1}{\cos \varepsilon} \| B_1 + T_1 \| \frac{1}{\gamma} \| B_2 + T_2 \| < 1$$

Then problem (1)-(2) is regularly solvable.

Corollary 2. If the conditions 1)-3) are fulfilled, and

$$q_1 = \frac{1}{2\sqrt{\gamma}} \frac{1}{\cos\varepsilon} \| B_1 \| \frac{1}{\gamma} \| B_2 \| < 1 \quad (3)$$

where $B_j = A_j A^{-j}$ $(j = 0,2)$, the problem (1), (2) is regularly solvable for $T_1 = 0$, $T_2 = 0$.

Theorem 4. Let the conditions 1)-4) and inequality (3) be fulfilled. Then problem (1)-(2) is Fredholm solvable.

IV. CONCULATION

As is known some authors have mainly studied the Cauchy problem. In a finite domain, boundary value problems with variable coefficients have been studied very little. We can note the works of S.S. Mirzoev. In our work here have considering boundary value problem operator differential equations. The paper shows how the regular and Fredholm solvability of the boundary value problem are related with the norms of the intermediate derivative operators. We hope that the results receiving here will be found over its following.

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REFERENS

[1]. Agayeva G.A. On a boundary value problem for operator-differential equations of the second order. Proceedings of the Pedagogical University, section of the natural sciences.p.9-17,(2017)

[2]. Agayeva G.A. On the existence and uniqueness of the generalized solution of a boundary value problem for second order operator-differential equations Transactions of NAS of Azerbaijan, №4, p. 3-8, (2014).

[3]. Agayeva G.A. On the solvability of a boundary value problem for elliptic operator-differential equations with an operator coefficient in the boundary condition. Bulletin of Baku University, №1, p.37-42, (2015)

[4]. Aliyev A.R. Boundary Value Problems for a Class of High-Order Operator-Differential Equations with Variable Coefficients. Matem zametki, 74(6), 803-814 (2003).

[5]. Aliyev A.R. On the solvability of a class of boundary value problems for second order operator differential equations in high space. Differential urav. 43(10), (2007)

[6]Dezin A.A. General issues of theory of boundary value problem M:Nauka,1980,207 p.

[7] Dubinsky Yu.A. On some arbitrary order differentia operator equations. Matemat. sbornik, 90(1),(132),3-22(1972).

[8]. Gasımov M.G. On solvabilety of boundary value problem for a class operator-differential equation. DAN SSSR, 235(3),505-508 (1972).

[9].Gasimova G.M. On solvability conditions of a boundary value problem with an operator in the boundary condition for a second order elliptic operator-differential equations. Proceeding of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan Volume 40, p. 172-177, 2014

[10]Gasimova G.M. On well-defined solvability of a boundary value problem for an elliptic differential equation in Hilbert space. Transactions of National Academy of Sciences of Azerbajan, Series of Physical-Technical and Mathematical Sciences, Issuse Mathematics, Baku, 35(1), 31-36, 2015.

[11] Gorbacuk V.İ.,Gorbacuk M.L., Boundary value problem for differential-operator equations. Kiev,Nauova Dumka,284 p(1984)

[12].Lions-J.L., Magenes E. Inhomogeneous boundary value problems and their applications. Moscow: Mir, 1977, 371 p.

[13].Mirzoyev S.S., Aghayeva G.A. On the solvability conditions of solvability of one boundary value problems for the second order differential equations with operator coefficients. Inter. Journal of Math. Analyzis, p.149-156, 2014, v.8, №4.

1. [14].Mirzoyev S.S., Agayeva G.A. On correct solvability of one boundary value problems for the differential equations of the second order on Hilbert space. Applied Mathematical Sciences 2013,v.7, p.3935-3945.№79.

[15].Mirzoev S.S., Aliyev A.R., Gasimova G.M. Solvability conditions for a boundary value problem with operator coefficients and related estimations of the norms of intermediate derivatives. Dokl.RAN №2, p. 511-513, 2016.

[16]. Mirzoyev S.S., Aliyev A.R., Rustamova L.A. On the boundary value problem with operator in boundary conditions for the operator-differential equations of second order with discontinuous coefficients. Journal of Mathem. Physics, Analysis, Geometry, 2013, v.9, №2, p.207-206.

[17].Mirzoyev S.S., Aliyev A.R., Rustamova L.A. Solvability conditions for boundary value problem for elliptic operator-differential equations with discontinuers coefficient. Math. Zametki, V.92.P789-793(2012)

[18].Mirzoyev S.S., Conditions for well-defined solvability of boundary value problems for operator differential equations Mirzoyev S.S., DAN, SSSR, №2, p.291-295(1983)

[19]Skalikov A.A. Elliptic equations in Hilbert space and related problems. prof. I.G.Petrovsky seminars,14,140-224.

[20]Yakubov S.Ya. Linear-differential equations and their applications Baku, Elm, 220p.(1985)