Inverse Problem for a Burgers Type Parabolic Equation

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***Abstract*—The paper considers the inverse problem of determining the unknown coefficient on the right side of the Burger's types parabolic equation Direct problems for the Burger's equation are considered in the works L. K.Martinson, J.Whitham and others. An additional condition for finding the unknown coefficient, which depends on the variable time, is given in integral form. Theorem on the uniqueness and stability of the solution was proved.**

Keywords—inverse problem, parabolic equation of Burger's type, the unknown coefficient on the right side, Hadamard ill-posed problems, uniqueness, conditional stability

I. Introduction

The Burgers equation is a special case of the Navir-Stokes equation in the one-dimensional case oud reflects both the effects of nonlinear propagation and the effects of diffusion.

Burgers equation arises when considering a wide class of processes in hydromechanics, nonlinear acoustics and plasma physics.

Burgers equation with a source $F(x,t) \neq 0$ describes the dynamics of a physical system located in an external field and is a natural generalization of the homogeneous equation corresponding to autonorms notions.

In papers [1,2] (see also the bibliography in these papers) direct problems for the Burgers equation are considered.

II. Problem State

Let $D = (0,1) \times (0,T], 0 < T = const$, the spaces $C^l(\cdot), C^{l+\alpha}(\cdot), C^{l,l/2}(\cdot), C^{l+\alpha,(l+\alpha)/2}(\cdot), l = 0,1,2, 0 < \alpha < 1$ and the norms in these spaces are defined as in [3,chapter 1]:

$$\|p\|_D^{(l)} = \sum_{k=0}^{l} \sup_D \left| \frac{\partial^k p(x,t)}{\partial x^k} \right|, \|q\|_T^{(l)} = \sum_{k=0}^{l} \sup_{[0,T]} \left| \frac{d^k q(t)}{dt^k} \right|.$$

We consider the following inverse problem on determining a pair of functions $\{f(t), u(x,t)\}$:

$$u_t - u_{xx} + uu_x = f(t)g(x), \; (x,t) \in D, \tag{1}$$

$$u(x,0) = \varphi(x), \quad x \in [0,1], \tag{2}$$

$$u(0,t) = \psi_0(t), \; u(1,t) = \psi_1(t), \; t \in [0,T], \tag{3}$$

$$\int_0^1 u(x,t)dx = h(t), \; t \in [0,T], \tag{4}$$

where $g(x), \varphi(x), \psi_0(t), \psi_1(t), h(t)$ are the given functions,

$$u_t = \frac{\partial u}{\partial t}, u_x = \frac{\partial u}{\partial t}, u_{xx} = \frac{\partial^2 u}{\partial x^2}.$$

The coefficient inverse problem for a parabolic equation were studied in the papers [4,5].

Problem (1)-(4) belugas to the class of Hadamard ill-posed problems.

Examples can be given that show that the solution to problem (1)-(4) does not always exist, and if it does, it way not be unique or unstable.

Therefore, this problem should be treated proceeding from the general concepts of the theory of ill-posed problems.

We wake the following assumptions for the data of problem (1)-(4):

1^0. $g(x) \in C^\alpha[0,1], \int_0^1 g(x)dx = g_0 \neq 0;$

2^0. $\varphi(x) \in C^{2+\alpha}[0,1];$

3^0. $\psi_0(t), \psi_1(t) \in C^{1+\alpha}[0,T], \varphi(0) = \psi_0(0), \varphi(1) = \psi_1(0);$

4^0. $h(t) \in C^{1+\alpha}[0,T].$

Definition 1. The pair of functions $\{f(t), u(x,t)\}$ is called the solution of problem (1)-(4) if:

1) $f(t) \in C^\alpha[0,T];$

2) $u(x,t) \in C^{2+\alpha,1+\alpha/2} \cap C(\overline{D});$

3) the conditions (1)-(4) hold these functions.

Problem (1)-(4) is reduced to an equivalent problem.

Lemma 1. Let the initial data of problem (1)-(4) satisfy conditions 1^0-4^0 and $\int_0^1 \varphi(x)dx = h(0)$.

If the pair $\{f(t),u(x,t)\}$ is a solution to problem (1)-(4) in the sense of definition 1, then this pair is also a solution to problem (1), (2), (3) and

$$f(t)=\left[\ h_t-u_x(1,t)+u_x(0,t)+\frac{\psi_1^2(t)}{2}-\frac{\psi_0^2(t)}{2}\right]\Big/g_0,\ t\in[0,T] \tag{5}$$

and vice versa, the solution to problem (1),(2),(3), (5) in the sense of definition 1 is a solution to problem (1)-(4).

Proof. Let's assume $\{f(t),u(x,t)\}$ is a solution to the problem in the sense of definition 1. Integrate equation 1 in the interval (0,1) with respect to x. Taking into account the conditions of lemma 1, we obtain

$$\int_0^1 u_t dx-\int_0^1 u_{xx}dx+\int_0^1 uu_x dx=\int_0^1 f(x)g(x)dx. \tag{6}$$

From here we obtain formula (5).

Now let's assume that the pair $\{f(t),u(x,t)\}$ is a solution to problem (1),(2),(3), (5) in the sense of definition 1. Let's show that conditions (4) are satisfied. If the ratio (6) together with $f(t)$, we write the value of (5) and denote $\theta(t)=\int_0^1 u(x,t)dt-h(t),$ we obtain

$$\theta_t(t)=0,\quad \theta(0)=0.$$

It is clear that the only solution to the Cauchy problem for $\theta(t)$ is $\theta(t)=0.$ Hence it follows: $\int_0^1 u(x,t)dx-h(t)=0.$

The Lemma 1 is proved.

Let us call problem (1),(2),(3) and (5) problem A.

III. On the uniqueness and "conditional" stability of a solution

The uniqueness theorem and the estimate of stability for the solution of inverse problem occupy a central place in investigation of their well-posedness.

Define the following set:

$K_\alpha=\{(f,u)/\,f(t)\in C^\alpha[0,T],u(x,t)\in C^{2+\alpha,1+\alpha/2}(\overline{D}),$
$(x,t)\in\overline{D}$ for all (f,u) $|f(t)|\le c_1,t\in[0,T],|u|,|u_x|,$

$|u_{xx}|\le c_2,(x,t)\in\overline{D},\ 0<c_1,c_2=const\}$

Let us assume that the two input sets $\{g_i(x),\varphi_i(x),\psi_{0i}(t),\ \psi_{1i}(t),h_i(t)\},i=1,2$.

Problem A for brevity of the further exposition, Problem A1 will call Problem A2 . Let $\{f_1(t),u_1(x,t)\}$ and $\{f_2(t),u_2(x,t)\}$ be solutions of problems A1 and A2 respectively.

Theorem 1. Let the following conditions hold

1) the function $\{g_i(x),\varphi_i(x),\psi_{0i}(t),$ $\psi_{1i}(t),h_i(t)\}$, $i=1,2$.

satisfy conditions 1^0-4^0, respectively;

2) solutions of problems A1 and A2 exist in the sense of definition 1 and they belong to the set K_α.

Then there exists a $T^*(0<T^*\le T)$, such that for $(x,t)\in[0,1]\times[0,T^*]=\overline{D_*}$ the solution of problem A is unique, and the stability estimate

$$\begin{aligned}&\|u_1-u_2\|_D^{(0)}+\|f_1-f_2\|_T^{(0)}\le\\&\le c_3\Big[\|g_1-g_2\|_{[0,1]}^{(0)}+\|\varphi_1-\varphi_2\|_{[0,1]}^{(0)}+\\&+\|\psi_{01}-\psi_{02}\|_T^{(1)}+\|\psi_{11}-\psi_{12}\|_T^{(1)}+\|h_1-h_2\|_T^{(1)}\end{aligned} \tag{7}$$

Proof. Consider the function

$$\begin{aligned}F(x)&=\varphi(x)+(1-x)[\psi_0(t)-\psi_0(0)]+\\&+x[\psi_1(t)-\psi_1(0)],(x,t)\in\overline{D}.\end{aligned}$$

It's obvious that $F(x,t)\in C^{2+\alpha,1+\alpha/2}(\overline{D})$ and $F(x,0)=\varphi(x),\quad x\in[0,1],F(0,t)=\psi_0(t),F(1,t)=\psi_1(t),$ $t\in[0,T]$.

Denote

$$\begin{aligned}&z(x,t)=u_1(x,t)-u_2(x,t),\lambda(t)=f_1(t)-f_2(t),\\&\delta_1(x)=g_1(x)-g_2(x),\delta_2(x)=\varphi_1(x)-\varphi_2(x),\\&\delta_3(t)=\psi_{01}(t)-\psi_{02}(t),\delta_4(t)=\psi_{11}(t)-\psi_{12}(t),\\&\delta_5(x)=h_1(t)-h_2(t),\delta_6(x,t)=F_1(x,t)-F_2(x,t),\\&w(x,t)=z(x,t)-\delta_6(x,t).\end{aligned}$$

Subtracting relations from the relations of problem A1 the corresponding relations of problem A2 we obtain the problem of determining a pair $\{\lambda(t),w(x,t)\}$:

$$w_t-w_{xx}=\lambda(t)g_1(x)+\Phi(x,t),\ (x,t)\in D, \tag{8}$$

$$w(x,0)=0,x\in[0,1], \tag{9}$$

$$w(0,t)=w(1,t)=0,t\in[0,T], \tag{10}$$

$$\lambda(t)=[z_x(0,t)-z_x(1,t)]/g_{0,1}+H(t),\ t\in[0,T], \tag{11}$$

where

$$\begin{aligned}&\Phi(x,t)=f_2(t)\delta_1(x)-z(x,t)u_{1x}(x,t)-z_x(x,t)u_2(x,t)-\\&-\delta_{6t}(x,t)+\delta_{6xx}(x,t),\\&H(t)=\{[\delta_{5t}(t)+\delta_4(t)(\psi_{11}(t)+\psi_{12}(t))/2-\\&-\delta_3(t)(\psi_{01}(t)+\psi_{02}(t)/2]g_{01}-\delta_1(x)[h_{2t}(t)-u_{2x}(1,t)+\\&+u_{2x}(0,t)+\psi_{12}^2(t)/2-\psi_{02}^2(t)/2]\}/g_{01}g_{0x}.\end{aligned}$$

Under the conditions of theorem 1 and from the definition of the set K_α it follows that right side of equation (8) satisfy the Holder's condition. It means, there exist a classical solution of definition problem of $w(x,t)$ from the conditions (8),(9),(10) and it can be represented in the form [3, p.468]:

$$w(x,t)=\int_0^t\int_0^1 G(x,t,\xi,\tau)\left[\lambda(\tau)g_1(\xi)+\Phi(\xi,\tau)\right]d\xi d\tau, \quad (12)$$

where $G(x,t,\xi,\tau)$ is fundamental solution to equation $u_t-u_{xx}=0$, for which the following estimations [3,chapter IV] are true:

$$\int_0^1\left|G_x^l(x,t,\xi,\tau)\right|d\xi\le c_4(t-\tau)^{-(l-\alpha)/2}, l=0,1 \quad (13)$$

where $0<c_4=const.$

Taking into account that $w(x,t)=z(x,t)-\delta_6(x,t)$, from (12) we will obtain:

$$z(x,t)=\delta_6(x,t)+ \\ +\int_0^t\int_0^1 G(x,t,\xi,\tau)\left[\lambda(\tau)g_1(\xi)+\Phi(\xi,\tau)\right]d\xi d\tau, \quad (14)$$

Assume

$$\chi=\|u_1-u_2\|_D^{(0)}+\|f_1-f_2\|_T^{(0)}.$$

Under the conditions of theorem 1 and from definition of the set K_α, taking into account estimation (13) we will obtain:

$$|z(x,t)|\le c_5\left[\|\delta_1\|_{[0,1]}^{(0)}+\|\delta_6\|_D^{(2,1)}\right]+c_6\chi t^{1+\alpha/2}+ \\ +c_7\|z_x\|_D^{(0)}t^{1+\alpha/2}, (x,t)\in\overline{D}, \quad (15)$$

$$|\lambda(t)|\le c_8\left[\|\delta_1\|_{[0,1]}^{(0)}+\|\delta_3\|_T^{(0)}+\|\delta_4\|_T^{(0)}+\|\delta_5\|_T^{(0)}\right]+ \\ +c_9\|z_x\|_D^{(0)}, \quad t\in[0,T]. \quad (16)$$

Function $z_x(x,t)$ estimated as follows: from formula (15) we have:

$$z_x(x,t)=\delta_{6x}(x,t)+ \\ +\int_0^t\int_0^1 G_x(x,\xi,t,\tau)\left[\lambda(\tau)g_1(\xi)+\Phi(\xi,\tau)\right]d\xi d\tau$$

under the conditions of Theorem 1, from the last equality we obtain

$$\|z_x(x,t)\|\le c_{10}\left[\|\delta_1\|_{[0,1]}^{(0)}+\|\delta_6\|_{[0,1]}^{(2,1)}\right]+c_{11}\chi t^{1+\alpha/2}+ \\ +c_{12}\|z_x\|_D^{(0)}t^{1+\alpha/2}, (x,t)\in\overline{D}, \quad (17)$$

where $c_{10},c_{11},c_{12}>0$ depends on data of problem A1 and A2 the set K_α.

Inequalities (17) are satisfied at one values $(x,t)\in\overline{D}$. Therefore they must be satisfied also for maximum values of the left parts.

$$\|z_x\|_D^{(0)}\le c_{10}\left[\|\delta_1\|_{[0,1]}^{(0)}+\|\delta_6\|_D^{(2,1)}\right]+c_{11}\chi t^{1+\alpha/2}+ \\ +c_{12}\|z_x\|_D^{(0)}t^{1+\alpha/2}. \quad (18)$$

Let $T_1(0<T_1\le T)$ be such numbers, that $c_{12}T_1^{1+\alpha/2}<1$. Then from (18) we have

$$\|z_x\|_D^{(0)}\le c_{13}\left[\|\delta_1\|_{[0,1]}^{(0)}+\|\delta_6\|_D^{(2,1)}\right]+c_{14}\chi t^{1+\alpha/2}.$$

Substituting the last inequality into (15) and (16) we obtain:

$$|z(x,t)|\le c_{15}\left[\|\delta_1\|_{[0,1]}^{(0)}+\|\delta_6\|_D^{(2,1)}\right]+c_{16}\chi t^{1+\alpha/2}, (x,t)\in\overline{D}$$

$$|\lambda(t)|\le c_{17}\left[\|\delta_1\|_{[0,1]}^{(0)}+\|\delta_6\|_D^{(2,1)}+\|\delta_5\|_T^{(0)}\right]+c_{18}\chi t^{1+\alpha/2}, t\in[0,T].$$

or

$$|z(x,t)|+|\lambda(t)|\le c_{19}\left[\|\delta_1\|_{[0,1]}^{(0)}+\|\delta_6\|_D^{(2,1)}+\|\delta_5\|_T^{(0)}\right]+c_{18}\chi t^{1+\alpha/2}.$$

The last inequality is satisfied for all $(x,t)\in\overline{D}$. therefore it is also satisfied for the maximum values of the left points.

$$\chi\le c_{19}\left[\|\delta_1\|_{[0,1]}^{(0)}+\|\delta_6\|_D^{(2,1)}+\|\delta_5\|_T^{(0)}\right]+c_{18}\chi t^{1+\alpha/2}.$$

Let $T_2(0<T_2\le T)$ R be such number, that $c_{18}T_1^{1+\alpha/2}<1$. Choosing $T=\min(T_1,T_2)$ from the last inequality we get the estimate (7).

The uniqueness of the solution to problem A is obtained from formula (7) at $g_1(x)=g_2(x), \varphi_1(x)=\varphi_2(x)$

$\psi_{01}(t)=\psi_{02}(t)$ $\psi_{11}(t)=\psi_{12}(t)$ $h_1(t)=h_2(t)$.
Theorem 1 is proved.

References

[1] L.K.Martiuson, Yu. J. Malov, Diffetential equations of mathematical physics. Moskow 2011 (in Russian)

[2] J.Whithow, Linear and nonlinear varies worl 1977 (In Russian)

[3] O.A.Ladyzhenskaya, V.A.Solonnikov and N.N.Uraltseva, Linear and Quasilinear equations of parabolic type [in Russian] Nauka Moscow (1967)

[4] A.Ya.Akhundov, A.J.Gasanova, Determination of the coefficient of a semilinear parabolic equation in the case of boundary-value problem with nonlinear boundary condition. Ukrainan Mathematical Journal 2017 v.66, N 6, p 847-852.

[5] A.Ya.Akhundov, A.Habibiova, On one inverse problem for a parabolic equation in domain with moving boundaries. Proceedings of the Just.of Math. And Mech., Nat.acad.of Sci. of Azerb., 2021, v.47, N 2,p262-269.