Some Spectral Properties of the One-Dimensional Schrödinger Operator with Additional Potential

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***Abstract*—One of the main study areas in quantum scattering theory is the one-dimensional Schrödinger operator. For the past two centuries a substantial number of mathematicians have made contribution to the study of this subject. The paper at hand considers a one-dimensional Schrödinger equation with additional potential on the entire axis. Additional potential is of the form $\theta(x)x^2$, where $\theta(x)$ is the Heaviside step function. It is well known that the solutions to such equation are expressed by Weber function. By matching the solutions at $x = 0$, representations of special solutions to the equation are obtained. Expansion formulas for these special solutions are then given using methods of Titchmarsh.**

***Keywords*—Schrödinger equation, Weber function, expansion formulas, reflection coefficient, scattering problem**

Consider the following Schrödinger equation:

$$-u'' + \frac{x^2}{4}u = -au, \qquad -\infty < x < \infty. \quad (1)$$

It is well known that ([1],[2]) this equation admits a solution $u = D_{-a-\frac{1}{2}}(x)$, which satisfies the initial conditions:

$$D_{-a-\frac{1}{2}}(0) = \frac{\sqrt{\pi}}{2^{\frac{2a+1}{4}}\Gamma\left(\frac{3}{4}+\frac{a}{2}\right)},$$
$$D'_{-a-\frac{1}{2}}(0) = -\frac{\sqrt{\pi}}{2^{\frac{2a-1}{4}}\Gamma\left(\frac{1}{4}+\frac{a}{2}\right)}. \quad (2)$$

Here, Γ denotes Euler's Gamma function, and $D_\nu(x)$ is the Weber function.

Let us now introduce the substitution $x \to \sqrt{2}x + \frac{1}{\sqrt{2}}$ into equation (1), defining a new function $y(x) = u\left(\sqrt{2}x + \frac{1}{\sqrt{2}}\right)$. This leads to the following relations for the derivatives of $y(x)$:

$$y'(x) = \sqrt{2}u'\left(\sqrt{2}x + \frac{1}{\sqrt{2}}\right), y''(x) = 2u''\left(\sqrt{2}x + \frac{1}{\sqrt{2}}\right)$$

Substituting these into equation (1) yields:

$$-\frac{y''}{2} + \frac{x^2+x}{2}y + \frac{1}{8}y = -ay.$$

Assuming $\lambda = -2a - \frac{1}{4}$, this simplifies to the equation:

$$-y'' + (x^2+x)y = \lambda y, -\infty < x < \infty. \quad (3)$$

Using the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, the solution to equation (3) is given by:

$$\psi(x,\lambda) = D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\sqrt{2}x + \frac{1}{\sqrt{2}}\right),$$

which satisfies the following conditions:

$$\psi(0,\lambda) = D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) = 2^{\frac{4\lambda-3}{16}}\frac{\sqrt{\pi}}{\Gamma\left(\frac{11}{16}-\frac{\lambda}{4}\right)}$$
$$= \cos\frac{(4\lambda-3)\pi}{16}\frac{2^{\frac{4\lambda-3}{16}}}{\sqrt{\pi}}\Gamma\left(\frac{\lambda}{4}+\frac{5}{16}\right), \quad (4)$$

$$\psi'(0,\lambda) = D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) = -2^{\frac{4\lambda+5}{16}}\frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{16}-\frac{\lambda}{4}\right)}$$
$$= \sin\frac{(4\lambda-3)\pi}{16}\frac{2^{\frac{4\lambda+5}{16}}}{\sqrt{\pi}}\Gamma\left(\frac{\lambda}{4}-\frac{13}{16}\right). \quad (5)$$

Similarly, if we substitute $y(x) = u\left(-\sqrt{2}x - \frac{1}{\sqrt{2}}\right)$ into equation (1) and assume $\lambda = -2a - \frac{1}{4}$, we obtain the same equation (3) and the solution

$$\varphi(x,\lambda) = D_{\frac{\lambda}{2}-\frac{3}{8}}\left(-\sqrt{2}x - \frac{1}{\sqrt{2}}\right),$$

which satisfies the following conditions:

$$\varphi(0,\lambda) = D_{\frac{\lambda}{2}-\frac{3}{8}}\left(-\frac{1}{\sqrt{2}}\right) = 2^{\frac{4\lambda-3}{16}}\frac{\sqrt{\pi}}{\Gamma\left(\frac{11}{16}-\frac{\lambda}{4}\right)}$$
$$= \cos\frac{(4\lambda-3)\pi}{16}\frac{2^{\frac{4\lambda-3}{16}}}{\sqrt{\pi}}\Gamma\left(\frac{\lambda}{4}+\frac{5}{16}\right), \quad (6)$$

$$\varphi'(0,\lambda) = D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(-\frac{1}{\sqrt{2}}\right) = -2^{\frac{4\lambda+5}{16}}\frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{16}-\frac{\lambda}{4}\right)}$$
$$= \sin\frac{(4\lambda-3)\pi}{16}\frac{2^{\frac{4\lambda+5}{16}}}{\sqrt{\pi}}\Gamma\left(\frac{\lambda}{4}-\frac{13}{16}\right). \quad (7)$$

Now, consider the equation:

$$-y'' + \theta(x)(x^2 + x)y = \lambda y, -\infty < x < \infty \quad (8)$$

where $\theta(x)$ is the Heaviside step function. Our goal is to find special solutions of this equation. Let G denote the complex λ-plane, with its positive semi-axis cut off. We denote by ∂G the lower and upper boundaries along this semi-axis. We define the function $\sqrt{\lambda}$ by selecting an analytic branch such that $\sqrt{\lambda + i0} > 0$ for $\lambda > 0$, and $\sqrt{\lambda - i0} < 0$ for $\lambda < 0$.

For $x \geq 0$, equation (8) is equivalent to equation (3), for which we have already determined solutions. For $x \leq 0$, the equation becomes a second-order linear differential equation with constant coefficients, whose general solution is

$$y = c_1 e^{i\sqrt{\lambda}x} + c_2 e^{i\sqrt{\lambda}x}. \quad (9)$$

Since $e^{i\sqrt{\lambda}x}$ and $e^{-i\sqrt{\lambda}x}$ are linearly independent for $\lambda \neq 0$, any solution for $x \leq 0$ can be expressed as a linear combination of these two functions. We now match this solution with $D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\sqrt{2}x + \frac{1}{\sqrt{2}}\right)$ at $x = 0$, requiring that both the solutions and their first order derivatives coincide at this point. This yields the system of equations:

$$\begin{cases} c_1 + c_2 = D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) \\ i\sqrt{\lambda}c_1 - i\sqrt{\lambda}c_2 = \sqrt{2}D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) \end{cases}.$$

Solving this system, we obtain:

$$c_1 = \frac{1}{2}\left[D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) - i\sqrt{\frac{2}{\lambda}}D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right],$$
$$c_2 = \frac{1}{2}\left[D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{\frac{2}{\lambda}}D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right].$$

On the other hand, for $x > 0$, the functions $D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\sqrt{2}x + \frac{1}{\sqrt{2}}\right)$ and $D_{\frac{\lambda}{2}-\frac{3}{8}}\left(-\sqrt{2}x - \frac{1}{\sqrt{2}}\right)$ represent linearly independent solutions to equation (8) for values of $\lambda \neq 2n + \frac{3}{4}$, where n is a non-negative integer. Therefore, for $x > 0$, any solution to equation (8) can be expressed as a linear combination of these two functions. Additionally, it is evident that for $x \leq 0$, the function $e^{-i\sqrt{\lambda}x}$ serves as a solution to the equation. In a similar manner, these solutions are smoothly connected at $x = 0$ as follows:

$$\begin{cases} a_1 D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) + a_2 D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) = 1 \\ a_1\sqrt{2}D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) - a_2\sqrt{2}D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) = -i\sqrt{\lambda} \end{cases}$$

Solving this system of equations yields the following coefficients:

$$a_1 = \frac{1}{2}\left[D^{-1}{}_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) - i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right)^{-1}\right],$$
$$a_2 = \frac{1}{2}\left[D^{-1}{}_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right)^{-1}\right].$$

Thus, we establish the following theorem:

Theorem 1. For any complex value of the parameter λ, the equation (8) has solutions $\phi_{\pm}(x, \lambda)$ that can be expressed as:

$$\phi_+(x,\lambda) = \begin{cases} D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\sqrt{2}x + \frac{1}{\sqrt{2}}\right), & x \geq 0 \\ \frac{1}{2}\left[D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) - i\sqrt{\frac{2}{\lambda}}D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right] \times \\ \times e^{i\sqrt{\lambda}x} + \\ +\frac{1}{2}\left[D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{\frac{2}{\lambda}}D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right] \times \\ \times e^{-i\sqrt{\lambda}x}, & x < 0. \end{cases} \quad (10)$$

$$\phi_-(x,\lambda) =$$

$$= \begin{cases} \frac{1}{2}\left[D^{-1}{}_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) - i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right)^{-1}\right] \times \\ \times D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\sqrt{2}x + \frac{1}{\sqrt{2}}\right) + \\ +\frac{1}{2}\left[D^{-1}{}_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right)^{-1}\right] \times \\ \times D_{\frac{\lambda}{2}-\frac{3}{8}}\left(-\sqrt{2}x - \frac{1}{\sqrt{2}}\right), & x \geq 0 \\ e^{-i\sqrt{\lambda}x}, & x < 0. \end{cases} \quad (11)$$

Note that for $\lambda \in \partial G$, formula (10) implies that the solution $\phi_+(x, \lambda)$ takes only real values. Moreover, if $\lambda > 0$, it follows from formula (11) that the function:

$$\overline{\phi_-(x,\lambda)} =$$

$$
= \begin{cases} \frac{1}{2}\left[D^{-1}{}_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right)^{-1}\right] \times \\ \times D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\sqrt{2}x + \frac{1}{\sqrt{2}}\right) + \\ + \frac{1}{2}\left[D^{-1}{}_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) - i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)\right)^{-1}\right] \times \\ \times D_{\frac{\lambda}{2}-\frac{3}{8}}\left(-\sqrt{2}x - \frac{1}{\sqrt{2}}\right), \quad x \geq 0 \\ e^{-i\sqrt{\lambda}x}, \quad x < 0. \end{cases} \quad (12)
$$

is also a solution to equation (8).

Next, we calculate the Wronskian of the solutions $\phi_-(x,\lambda)$ and $\overline{\phi_-(x,\lambda)}$. Since the Wronskian is independent of x, we have:

$$
W\{\phi_-(x,\lambda), \overline{\phi_-(x,\lambda)}\} = e^{-i\sqrt{\lambda}x} i\sqrt{\lambda} e^{i\sqrt{\lambda}x} - \\ - \left(-i\sqrt{\lambda}e^{-i\sqrt{\lambda}x}e^{i\sqrt{\lambda}x}\right) = 2i\sqrt{\lambda}. \quad (13)
$$

Thus, for $\lambda > 0$, these solutions are linearly independent, implying that $\phi_+(x,\lambda)$ can be expressed as a linear combination of $\phi_-(x,\lambda)$ and $\overline{\phi_-(x,\lambda)}$:

$$
\phi_+(x,\lambda) = a(\lambda)\overline{\phi_-(x,\lambda)} + \overline{a(\lambda)}\phi_-(x,\lambda), \lambda > 0. \quad (14)
$$

From these relations, we obtain the following expression for the Wronskian:

$$
W\{\phi_-(x,\lambda), \phi_+(x,\lambda)\} = 2i\sqrt{\lambda}a(\lambda).
$$

Thus, the coefficient $a(\lambda)$ can be determined as:

$$
a(\lambda) = \frac{1}{2}D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2\lambda}}D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right). \quad (15)
$$

We now proceed to derive the expansion formulas for the solutions to equation (8). To this end, we define the following functions:

$$
\rho(\lambda) = \begin{cases} 0, \lambda \leq 0 \\ \int_0^\lambda \frac{|a(u)|^{-2}}{4\sqrt{u}} du, \lambda > 0 \end{cases}, \quad (16)
$$

$$
r(\lambda) = \frac{\overline{a(\lambda)}}{a(\lambda)} = \frac{\frac{1}{2}D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2\lambda}}D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{2}D_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2\lambda}}D'_{\frac{\lambda}{2}-\frac{3}{8}}\left(\frac{1}{\sqrt{2}}\right)}. \quad (17)
$$

The function $r(\lambda)$ is referred to as the reflection coefficient for the scattering problem associated with equation (8). From formula (14), we obtain the following relationship:

$$
\frac{\phi_+(x,\lambda)}{a(\lambda)} = \overline{\phi_-(x,\lambda)} + r(\lambda)\phi_-(x,\lambda), \lambda \in \partial G. \quad (18)
$$

Theorem 2. The following expansion formulas hold:

$$
\frac{1}{4\pi}\int_{\partial G} \frac{1}{\sqrt{\lambda}}\left[\overline{\phi_-(x,\lambda)} + r(\lambda)\phi_-(x,\lambda)\right]\phi_-(y,\lambda)d\lambda \\ = \delta(x-y), \quad (19)
$$

$$
\frac{1}{\pi}\int_{-\infty}^{+\infty} \phi_+(x,\lambda)\phi_+(y,\lambda)d\rho(\lambda) = \delta(x-y), \quad (20)
$$

where δ denotes the Dirac delta function.

Proof. Consider the equation:

$$
-y'' + (x^2 + x)\theta(x)y - \lambda y = f(x),
$$

where $f(x)$ is an arbitrary finite function that belongs to $L_{2,loc}$. Our goal is to derive an expansion formula for $f(x)$. Using the methods of Titchmarsh [3], we can express the Green's function in the following form:

$$
G(x,y,\lambda) = \begin{cases} \frac{\phi_+(x,\lambda)\phi_-(y,\lambda)}{2i\sqrt{\lambda}a(\lambda)}, y \leq x \\ \frac{\phi_-(x,\lambda)\phi_+(y,\lambda)}{2i\sqrt{\lambda}a(\lambda)}, y > x \end{cases}.
$$

Next, define the function:

$$
\Phi(x,\lambda) = \int_{-\infty}^{+\infty} G(x,y,\lambda)f(y)dy.
$$

It is well-known that the following relations hold:

$$
f(x) = -\frac{1}{i\pi}\int_{-\infty}^{+\infty} \Phi(x,\lambda+i0)d\lambda,
$$

$$
f(x) = \frac{1}{i\pi}\int_{-\infty}^{+\infty} \Phi(x,\lambda-i0)d\lambda
$$

From these, we deduce the following identity:

$$
f(x) = -\frac{1}{2\pi i}\int_{\partial G} \Phi(x,\lambda)d\lambda.
$$

Substituting the expressions for $\Phi(x,\lambda)$ and $G(x,y,\lambda)$ in this identity, we obtain:

$$
-\frac{1}{2\pi i}\int_{\partial G} \frac{\phi_+(x,\lambda)\phi_-(y,\lambda)}{2i\sqrt{\lambda}a(\lambda)}d\lambda = \delta(x-y), \quad (21)
$$

$$
-\frac{1}{2\pi i}\int_{\partial G} \frac{\phi_-(x,\lambda)\phi_+(y,\lambda)}{2i\sqrt{\lambda}a(\lambda)}d\lambda = \delta(x-y). \quad (22)
$$

Using formula (18), the left-hand side of (21) can be rewritten as:

$$\frac{1}{4\pi}\int_{\partial G}\frac{1}{\sqrt{\lambda}}\left[\overline{\phi_-(x,\lambda)}+r(\lambda)\phi_-(x,\lambda)\right]\phi_-(y,\lambda)d\lambda.$$

This relation leads to the expansion formula (19).
For the left-hand side of equation (22), we obtain the following expression:

$$-\frac{1}{2\pi i}\int_{\partial G}\frac{\phi_-(x,\lambda)\phi_+(y,\lambda)}{2i\sqrt{\lambda}a(\lambda)}d\lambda$$

$$=\frac{1}{4\pi}\int_0^{+\infty}\frac{\phi_-(x,\lambda+i0)\phi_+(y,\lambda+i0)}{\sqrt{\lambda+i0}a(\lambda+i0)}d\lambda+$$

$$+\frac{1}{4\pi}\int_{+\infty}^{0}\frac{\phi_-(x,\lambda-i0)\phi_+(y,\lambda-i0)}{\sqrt{\lambda-i0}a(\lambda-i0)}d\lambda=$$

$$=\frac{1}{4\pi}\int_0^{+\infty}\frac{\phi_+(y,\lambda)}{\sqrt{\lambda+i0}}\left[\frac{\phi_-(x,\lambda+i0)}{a(\lambda+i0)}+\frac{\overline{\phi_-(x,\lambda+i0)}}{\overline{a(\lambda+i0)}}\right]d\lambda.$$

Here, we used the fact that $\phi_+(y,\lambda)$ takes real values for $\lambda\in\partial G$, and that identities

$$\phi_-(x,\lambda-i0)=\overline{\phi_-(x,\lambda+i0)},a(\lambda-i0)=$$

$$=\overline{a(\lambda+i0)},\sqrt{\lambda-i0}=-\sqrt{\lambda+i0}$$

are true for $\lambda>0$. Using formula (14), we obtain:

$$\frac{\phi_+(x,\lambda)}{|a(\lambda)|^2}=\frac{\phi_-(x,\lambda+i0)}{a(\lambda+i0)}+\frac{\overline{\phi_-(x,\lambda+i0)}}{\overline{a(\lambda+i0)}}$$

for $\lambda>0$. Finally, by setting $\sqrt{\lambda+i0}=\sqrt{\lambda}$, we arrive at the expression:

$$-\frac{1}{2\pi i}\int_{\partial G}\frac{\phi_-(x,\lambda)\phi_+(y,\lambda)}{2i\sqrt{\lambda}a(\lambda)}d\lambda=$$

$$=\frac{1}{4\pi}\int_0^{+\infty}\frac{\phi_+(x,\lambda)\phi_+(y,\lambda)}{|a(\lambda)|^2\sqrt{\lambda}}d\lambda.$$

Taking into account the above equality and equation (16), we arrive at the expansion formula (20).
Thus, the theorem is proven.

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