Necessary Optimality Conditions in One Stepwise Nonsmooth Control Problem for Discrete Volterra-Type Systems

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***Abstract*—The paper considers the stepwise optimal control problem described by a difference analog of Volterra type integro-differential equation with a non-differentiable quality criterion. Optimal control problems for the stepwise processes are called optimal control problems for composite systems or multi-stage systems. In the problem under consideration, it is assumed that the right-hand side of the original equation and the quality functional have derivatives in any direction with respect to the state vector. First, using the increment method and some concepts of non-smooth analysis, a necessary optimality condition in terms of generalized directional derivatives is established. Then, based on it, necessary optimality conditions are established, directly expressed by the parameters of the non-smooth problem under consideration. The established result can be used to study the minimax problems for the different stepwise system.**

***Keywords*—Volterra type difference equation, stepwise control problem, non-smooth quality functional, directional derivative, optimality, minimax problem**

I. INTRODUCTION

As is known (see, for example, [1]), mathematical modeling of various problems in economics and technology leads to multi-stage optimization problems with non-smooth objective functionals. Such optimal control problems for the continuous systems are studied in works [2, 3], where additional bibliography is provided.

Along with this, discrete dynamic models of controlled systems are quite important from theoretical and practical points of view. There exits a class of such mathematical models that allow us to cover a very wide range of real objects and the optimal control problems corresponding to them.

Various aspects of optimal control problems described by the step discrete Volterra systems with smooth quality functionals have been studied in [4-6] and others.

In the proposed work, a similar problem is studied in the case of a non-smooth quality functional, while taking into account its specifics, the necessary optimality conditions are proved in terms of directional derivatives [7].

II. STATEMENT OF THE PROBLEM

Let $T_1=\{t_0,t_0+1,\dots,t_1-1\}, T_2=\{t_1,t_1+1,\dots,t_2-1\}$ be given finite sequences, where the difference t_2-t_0 is a natural number and $\Phi_1(x),\Phi_2(x)$ are given scalar functions satisfying the Lipschitz condition and having derivatives in any direction.

Let the controlled discrete process be described by the following system of nonlinear ordinary difference equations of the Volterra type

$$x(t+1)=\sum_{\tau=t_0}^{t} f(t,\tau,x(\tau),u(\tau)),\ \ t\in T_1,\ \ x(t_0)=x_0, \quad (1)$$

$$y(t+1)=\sum_{\tau=t_0}^{t} g(t,\tau,y(\tau),v(\tau)),$$
$$t\in T_2, y(t_1)=G\big(x(t_1)\big). \quad (2)$$

Here $f(t,\tau,x,u)\ \big(g(t,\tau,y,v)\big)$ is given $n-$dimensional vector function, continuous in $(x,u)\ \big((y,v)\big)$ with partial derivatives with respect to $x(y)$ for all (t,τ), $G(x)$ is given continuously differentiable $m-$ dimensional vector function.

If the control function $u(t)\big(v(t)\big)$ satisfies the constraint

$$u(t)\in U\subset R^r,\ \ t\in T_1, \quad (3)$$

$$v(t)\in V\subset R^q,\ \ t\in T_2,$$

then such a pair of control function $\big(u(t),v(t)\big)$ we will call an admissible control, and the corresponding process $\big(u(t),v(t),x(t),y(t)\big)-$ acceptable process.

Here $U(V)$ is given as a non-empty, bounded set.

The problem is to minimize the non-smooth functional

$$S(u,v)=\Phi_1(x(t_1))+\Phi_2(y(t_2)) \quad (4)$$

subject to (1)-(3).

The pair of admissible controls $(u(t),v(t))$ that provide the minimum value for functional (4) under constraints (1)-(3) will be called optimal control.

The purpose of this paper is to obtain first order necessary optimality conditions for in considered stepwise discrete non-smooth problem (1)-(4).

III. NECESSARY OPTIMALITY CONDITIONS IN TERMS OF DIRECTIONAL DERIVATIVES

Considering $(u(t), v(t), x(t), y(t))$ as fixed admissible process, we introduce the sets

$$f(t,\tau,x(\tau),U) = \{\alpha: \alpha = f(t,\tau,x(\tau),u), u \in U\},$$

$$g(t,\tau,y(\tau),V) = \{\beta: \beta = g(t,\tau,y(\tau),v), v \in V\}. \quad (5)$$

We assume that the sets (5) is convex for all (t,τ).

By $u(t;\varepsilon)(v(t;\varepsilon))$ we denote an arbitrary admissible control such that the corresponding state $x(t;\varepsilon)(y(t;\varepsilon))$ the process satisfies the relation:

$$x(t+1;\varepsilon) = \sum_{\tau=t_0}^{t} f(t,\tau,x(\tau;\varepsilon),u(\tau;\varepsilon)) =$$

$$= \sum_{\tau=t_0}^{t}(1-\varepsilon) f(t,\tau,x(\tau;\varepsilon),u(\tau)) + \sum_{\tau=t_0}^{t} \varepsilon f(t,\tau,x(\tau;\varepsilon),\breve{u}(\tau)), \quad (6)$$

$$y(t+1;\varepsilon) = \sum_{\tau=t_0}^{t} g(t,\tau,y(\tau;\varepsilon),v(\tau;\varepsilon)) =$$

$$= \sum_{\tau=t_0}^{t}(1-\varepsilon) g(t,\tau,y(\tau;\varepsilon),v(\tau)) + \sum_{\tau=t_0}^{t} \varepsilon g(t,\tau,y(\tau;\varepsilon),\breve{v}(\tau)), \quad (7)$$

where $\varepsilon \in [0,1]$ is an arbitrary number, and $\breve{u}(t) \in U,, t \in T_1, \breve{v}(t) \in V, t \in T_2$ are arbitrary admissible controls. Such a pair of admissible controls $u(t;\varepsilon), v(t;\varepsilon)$ exists due to the convexity of set (5).

Let by definition

$$z(t) = \left.\frac{\partial x(t;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad (8)$$

$$l(t) = \left.\frac{\partial y(t;\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}. \quad (9)$$

Using (6), (7) and taking into account the smoothness of the vector function $f(t,x,\tau,u)\big(g(t,y,\tau,v)\big)$ it follows that the vector function $z(t)(l(t))$ defined by formulas (8), (9), correspondingly are the solution of the following difference equation in Volterra type variations

$$z(t+1) = \sum_{\tau=t_0}^{t} f_x\big(t,\tau,x(\tau),u(\tau)\big)z(\tau) + \Big(f\big(t,\tau,x(\tau),\breve{u}(\tau)\big) - f\big(t,\tau,x(\tau),u(\tau)\big)\Big), \quad (10)$$

$$l(t+1) = \sum_{\tau=t_0}^{t} g_y\big(t,\tau,y(\tau),v(\tau)\big)l(\tau) + \Big(g\big(t,\tau,y(\tau),\breve{v}(\tau)\big) - g\big(t,\tau,y(\tau),v(\tau)\big)\Big). \quad (11)$$

On the other hand, from (7) and (8) it is clear that

$$x(t;\varepsilon) = x(t) + \varepsilon z(t) + o(\varepsilon;t),$$

$$y(t;\varepsilon) = y(t) + \varepsilon l(t) + o(\varepsilon;t).$$

In this case, a special increment of the quality functional (4), corresponding to the pair of admissible controls $(u(t;\varepsilon), v(t;\varepsilon))$ and $(u(t), v(t))$ can be written as:

$$\begin{aligned} S\big(u(t;\varepsilon), v(t;\varepsilon)\big) - S\big(u(t), v(t)\big) &= \Phi_1\big(x(t_1;\varepsilon)\big) - \Phi_1\big(x(t_1)\big) \\ &+ \Phi_2\big(y(t_2;\varepsilon)\big) - \Phi_2\big(y(t_2)\big) = \end{aligned}$$

$$\begin{aligned} &= \big[\Phi_1(x(t_1) + \varepsilon z(t_1) + o(t_1;\varepsilon)) - \Phi_1\big(x(t_1) + \varepsilon z(t_1)\big)\big] \\ &- \big[\Phi_1\big(x(t_1) + \varepsilon z(t_1)\big) - \Phi_1\big(x(t_1)\big)\big] + \\ &+ \big[\Phi_2(y(t_2) + \varepsilon l(t_2) + o(t_2;\varepsilon)) - \Phi_2\big(y(t_2) + \varepsilon l(t_2)\big)\big] \\ &- \big[\Phi_2\big(y(t_2) + \varepsilon l(t_2)\big) \\ &- \Phi_2\big(y(t_2)\big)\big]. \quad (12) \end{aligned}$$

Since the functions $\Phi_1(x)$, $\Phi_2(y)$ satisfy the Lipschitz condition, then it is clear that

$$|\Phi_1(x(t_1) + \varepsilon z(t_1) + o(\varepsilon)) - \Phi_1(x(t_1) + \varepsilon z(t_1))| \le o_1(\varepsilon),$$

$$|\Phi_2(y(t_2) + \varepsilon l(t_1) + o(\varepsilon)) - \Phi_2(y(t_2) + \varepsilon y(t_2))| \le o_1(\varepsilon).$$

From here, using the definition of the directional derivative, we obtain that

$$\Phi_1\big(x(t_1) + \varepsilon z(t_1)\big) - \Phi_1\big(x(t_1)\big) + \Phi_2\big(y(t_2) + \varepsilon l(t_2)\big) - \Phi_2\big(y(t_2)\big) =$$

$$= \varepsilon\left(\frac{\partial \Phi_1(x(t_1))}{\partial z(t_1)} + \frac{\partial \Phi_2(y(t_2))}{\partial l(t_2)}\right) + o_2(\varepsilon).$$

Now suppose that the admissible control $(u(t), v(t))$ is optimal. Then from the last expansion we obtain

$$\varepsilon\left(\frac{\partial \Phi_1(x(t_1))}{\partial z(t_1)} + \frac{\partial \Phi_2(y(t_2))}{\partial l(t_2)}\right) + o_2(\varepsilon) \ge 0.$$

Thus, the following theorem proved.

Theorem 1. If the set (5) is convex, then for the optimality of the admissible control $(u(t), v(t))$ in considered stepwise control problem (1)-(4) it is necessary that the inequality

$$\frac{\partial \Phi_1(x(t_1))}{\partial z(t_1)} + \frac{\partial \Phi_2(y(t_2))}{\partial l(t_2)} \ge 0 \quad (13)$$

is satisfied for all $u(t) \in U, t \in T_1, v(t) \in V, t \in T_2$.

Inequality (13) is an implicit criterion of optimality with respect to the parameters of problem (1)-(4). Using it, it is possible to obtain the necessary optimality condition expressed directly through the parameters of problem (1)-(4).

Equations (10), (11) are linear non-homogeneous difference equations. Therefore, applying the results of works [4, 8], we obtain that the solution $(z(t), l(t))$ of the equation in variations admits the representations

$$z(t)=\sum_{\tau=t_0}^{t-1}\sum_{s=\tau}^{t-1}R_1(t,s+1)\big(f(t,\tau,x(\tau),\breve{u}(\tau))-f(t,\tau,x(\tau),u(\tau)\big), \qquad (14)$$

$$l(t)=\sum_{\tau=t_1}^{t-1}\sum_{s=\tau}^{t-1}R_2(t,s+1)\big(g(t,\tau,y(\tau),\breve{v}(\tau))-g(t,\tau,y(\tau),v(\tau))+$$

$$+\sum_{\tau=t_0}^{t_1-1}\sum_{s=\tau}^{t_1-1}R_2(t,t_1)G_x(x(t_1))\,R_1(t_1,s+1)\Big(f(s,\tau,x(\tau),\breve{u}(\tau))-f(s,\tau,x(\tau),u(\tau))\Big). \qquad (15)$$

Here $R_1(t,s)$ $(R_2(t,s))$ is a matrix Volterra difference equation and is a resolvent of equation (10), ((11))

$$R_1(t+1,s)=\sum_{s=\tau}^{t}R_1(t+1;s+1)\,f_x(s,\tau,x(\tau),u(\tau))y(\tau),\quad \tau\le t,$$

$$R_1(t+1;t+1)=E_1,$$

$$R_2(t+1,\tau)=\sum_{s=\tau}^{t}R_2(t+1,s+1)\,g_y(s,\tau,x(\tau),u(\tau))l(\tau),\quad \tau\le t,$$

$$R_2(t+1,t+1)=E_2,$$

where E_1,E_2 are $(n\times n)$- dimensional identity matrices.

Taking into account (14), (15) we set

$$L_1(\breve{u})=\sum_{\tau=t_0}^{t_1-1}\left[\sum_{s=\tau}^{t_1-1}R(t_1,s+1)(f(s,\tau,x(\tau),v(\tau))-f(s,\tau,x(\tau),u(\tau))\right],$$

$$L_2(\breve{v})=\sum_{\tau=t_1}^{t_2-1}\sum_{s=\tau}^{t_2-1}R_2(t_2,s+1)\big(g(t_2,\tau,y(\tau),\breve{v}(\tau))-g(t_2,\tau,y(\tau),v(\tau))+$$

$$+\sum_{\tau=t_0}^{t_1-1}\sum_{s=\tau}^{t_1-1}R_2(t_2,t_1)G_x(x(t_1))\,R_1(t_1,s+1)\Big(f(s,\tau,x(\tau),\breve{u}(\tau))-f(s,\tau,x(\tau),u(\tau))\Big).$$

Therefore, it is proved

Theorem 2. If the set (5) is convex, then for the optimality of the admissible control $(u(t),v(t))$ in considered stepwise control problem (1)-(4) it is necessary that the inequality

$$\frac{\partial\Phi_1(x(t_1))}{\partial L_1(\breve{u})}+\frac{\partial\Phi_2(x(t_1))}{\partial L_2(\breve{v})}\ge 0$$

is satisfied for all $u(t)\in U, t\in T_1, v(t)\in V, t\in T_2$.

IV. Conclusion

In the proposed work, a non-smooth optimal control problem for the discrete Volterra systems is considered. Using the explicit linearization method, first order necessary optimality condition in terms of directional derivatives is established. The obtained results analyzed in Matlab.

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