A Mathematical Representation of the Collatz Conjecture and its Applications

Fidan Nuriyeva
Dokuz Eylul University, Faculty of Science, Department of Computer Science
Izmir, Türkiye
Institute of Control Systems,
The Ministry of Science and Education of the Republic of Azerbaijan
Baku, Azerbaijan
0000-0001-5431-8506

Urfat Nuriyev
Karabakh University,
Khankendi, Azerbaijan
Institute of Control Systems,
The Ministry of Science and Education of the Republic of Azerbaijan
Baku, Azerbaijan
0000-0002-3337-5859

***Abstract*— The Collatz Conjecture is more than just a mathematical and scientific subject of study. In addition to being an important partner in such scientific research from the perspective of Computer Engineering and Computer Science, there are various application areas where the conjectural algorithm is utilized in operations. There are also studies in the fields of steganography, cryptology, data hiding, and watermarking where the Collatz Conjecture algorithm is employed.**

In this study, research has been conducted on the applications of the Collatz Conjecture, and some mathematical expressions provided by the Collatz Procedure have been presented. These expressions can be useful in the proof and applications of the Collatz Conjecture. Additionally, a more in-depth review of the current research directions and applications of this conjecture is provided.

***Keywords*— collatz conjecture, computer computational methods, copyright protection; digital watermarkin, cryptography, encryption**

I. Introduction

The Collatz Conjecture is a problem proposed by Lothar Collatz in 1937, which has gained significant popularity in the field of mathematics. According to the conjecture, "Starting from any positive integer, the numbers obtained through a specific function will always reach 1" [1].

If the given number n is odd, it is multiplied by 3 and then 1 is added; if n is even, it is divided by 2. This iteration continues until the number reaches 1.

Numerous studies have been conducted to prove the hypothesis, and prominent mathematicians have worked on it for long periods. The problem has been investigated from various perspectives. Among the theoretical studies, we can mention [2]. In [3], the problem is represented in the base-4 number system, and in [4], a graph representation of the problem is provided based on the results of [3].

Hungarian mathematician Paul Erdős stated, "Mathematics is not yet ready for problems of this type" [1]. One of the most significant recent studies on the Collatz Conjecture belongs to Terence Tao. In his 2019 paper, he used probability theory to prove that almost all Collatz trajectories are bounded by some function diverging to infinity [5].

Beyond being merely a mathematical hypothesis, this problem has found various applications in fields such as computer science, cryptography, and steganography. The applications of the Collatz Conjecture offer innovative solutions, particularly in areas like data security, encryption, and digital authentication.

Recent studies on the mathematical foundations and application areas of the conjecture provide new perspectives on its mathematical resolution and potential technological applications. For instance, integrating the Collatz Conjecture into encryption algorithms plays a crucial role in secure digital authentication systems and digital watermarking technologies.

Today, research on the Collatz Conjecture is not only limited to mathematical analysis and theoretical investigations but also explores how these theoretical studies can be adapted to technology.

II. Some Applications of the Collatz Conjecture

The Collatz Procedure, beyond being a mathematical curiosity, has found applications in various scientific and technological fields. Due to its iterative and deterministic nature, it has been explored in areas such as cryptography, steganography, data security, and complexity analysis. Below are some notable applications of the Collatz Procedure:

A. Cryptography and Data Security

The unpredictable yet structured nature of the Collatz sequence has made it an interesting candidate for cryptographic applications. Some encryption algorithms incorporate Collatz-like transformations to introduce complexity and unpredictability into cryptographic keys. This enhances security against brute-force attacks and improves the randomness of key generation processes [6-9].

B. Steganography and Watermarking

Steganography, the practice of hiding information within other data, can leverage Collatz sequences to encode messages. By embedding information within the iterative steps of the Collatz sequence, hidden messages can be efficiently stored and transmitted in digital media such as images or audio files. Similarly, digital watermarking techniques use the Collatz sequence to create robust, undetectable marks for copyright protection [6, 7].

C. Pseudo-Random Number Generation

The sequence produced by the Collatz Procedure exhibits chaotic behavior, making it a potential candidate for pseudo-random number generators (PRNGs). While not entirely random, its structure can be exploited to produce sequences that approximate randomness, which can be useful in simulations, statistical sampling, and cryptographic applications [10, 11].

D. Graph Theory and Network Analysis

Collatz sequences can be represented as directed graphs, where numbers serve as nodes and transitions between numbers act as edges. These graph structures help in studying computational complexity, network behavior, and even optimization problems. Researchers have investigated how the graph representation of Collatz sequences can model various computational problems [12, 13].

E. Machine Learning and Algorithm Optimization

Recent studies have explored the use of Collatz-like transformations in training machine learning models. By mapping data transformations through iterative procedures inspired by the Collatz sequence, researchers aim to improve algorithmic efficiency and data processing techniques [14, 15] .

F. Complexity Theory and Computation

The Collatz Conjecture remains an open problem in computational mathematics, leading to studies in computational complexity and algorithmic behavior. Understanding the iterative patterns in Collatz sequences can provide insights into undecidable problems, iterative algorithm efficiency, and the behavior of dynamical systems [5, 15, 16].

III. THE COLLATZ CONJECTURE

A. Problem Definition

Definition: Starting from any positive integer n, the iterations of the function $f(n)$ will always reach the number 1.

- If the given n is odd, it is multiplied by 3 and increased by 1.
- If n is even, it is divided by 2.
- The iteration continues until the number 1 is reached.

The function in this definition is expressed as follows:

$f: Z^+ \to Z^+$

$f_0 = n, n \neq 0$

$f_{i+1} = f_i/2$, if f_i is even number,

$f_{i+1} = 3f_i + 1$, if f_i is odd number.

The Collatz Conjecture predicts that for any positive integer n, the iterations of a specific function will eventually reach 1. The function operates as follows:

If n is odd, apply $n = 3n + 1$

If n is even, apply $n = n/2$

These operations are repeated until n reaches 1. Each iteration of this mathematical procedure illustrates how a given number progresses toward 1.

B. Collatz Function

In mathematical notation, the Collatz function can be expressed as follows:

$$C: Z^+ \to Z^+$$

$$C(n) = \begin{cases} \frac{n}{2}, & n \to n \equiv 0(mod\ 2) \\ 3n+1 & n \to n \equiv 1(mod\ 2) \end{cases}$$

$$F(n) = \left\{n, C(n), C\big(C(n)\big), C\Big(C\big(C(n)\big)\Big), \dots\right\}$$
$$= \{n, C(n), C^2(n), C^3(n), \dots\}$$

C. Collatz Procedure

The step-by-step process of the Collatz function is as follows:

1. Choose a positive integer.
2. If the number is even, divide it by 2.
3. If the number is odd, multiply it by 3, add 1, and then divide by 2.
4. Repeat the process until the result is 1.

For example, let's consider the number 5:

1. 5 is odd → 3(5)+1=16
2. 16 is even → 16/2=8
3. 8 is even → 8/2=4
4. 4 is even → 4/2=2
5. 2 is even → 2/2=1

Result: The number 5 reaches 1 in 5 steps.

Collatz Sequences for Numbers from 1 to 10

C(1) = {1},

C(2) = {2, 1} ,

C(3) = {3, 10, 5, 16, 8, 4, 2, 1},

C(4) = {4, 2, 1},

C(5) = {5, 16, 8, 4, 2, 1},

C(6) = {6, 3, 10, 5, 16, 8, 4, 2, 1},

C(7) = {7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1},

C(8) = {8, 4, 2, 1},

C(9) = {9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1},

C(10) = {10, 5, 16, 8, 4, 2, 1}

D. Conjecture Notation

For every $n \in Z^+$, there exists a $k \in Z^+$ such that $C^k(n) = 1$

$$\forall n \in Z^+, \exists\ C^k(n) = 1\ k \in Z^+$$

As of the year 2020, computers have verified the Collatz Conjecture for all starting values up to 2^{68} ($2^{68} \approx 2.95 \times 10^{20}$). In every case, the sequence eventually reached 1. These results support the validity of the conjecture; however, a formal mathematical proof has not yet been found.

For this reason, the Collatz Conjecture is classified as: A mathematical hypothesis that is believed to be true but has neither been proven nor disproven.

IV. Some Mathematical Notations for the Collatz Conjecture

In this section, a more in-depth mathematical analysis of the Collatz function is presented. These analyses provide important insights into the proof of the Collatz Conjecture.

In the following parts of our paper, we will disregard even numbers and perform operations only with odd numbers.

For example, if we consider the Collatz Procedure for $n = 7$, it typically reaches 1 in 16 steps:

C(7) = {7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5,
16, 8, 4, 2, 1}

However, if we represent this procedure using only odd numbers, we reach 1 in just 5 steps:

CO(7) = { 7, 11, 17, 13, 5, 1 }

Here, CO represents the subset of C that includes only odd numbers.

In this case, the following theorem holds:

A. *Theorem 1*

For every n, the following equation holds:

$$3^m \cdot \boldsymbol{n} + 3^{m-1} + 3^{m-2} \cdot 2^{k_{m-2}} + 3^{m-3} \cdot 2^{k_{m-3}} + \cdots + +3^2 \cdot\cdot 2^{k_2} + 3^1 \cdot 2^{k_1} + 3^0 \cdot 2^{k_0} = 2^k \quad (1)$$

where m represents the number of steps in $CO(n)$.

Below, the expansion of equation (1) is provided for all odd numbers in the range n = 1 to 27:

$$3^1 \cdot \mathbf{1} + 3^0 = 3 + 1 = 4 = 2^2$$

$$3^2 \cdot \mathbf{3} + 3^1 + 3^0 \cdot 2^1 = 9 \cdot 3 + 3 + 1 \cdot 2 = = 27 + 3 + 2 = = 32 = 2^5$$

$$3^1 \cdot \mathbf{5} + 3^0 = 15 + 1 = 16 = 2^4$$

$$3^5 \cdot \mathbf{7} + 3^4 + 3^2 \cdot 2 + 3^2 \cdot 2^2 + 3^1 \cdot 2^4 + 2^7 = = 243 \cdot 7 + 81 + 27 \cdot 2 + 9 \cdot 24 + 3 \cdot 16 + 128 = 1702 + 81 + 54 + 36 + 48 + 128 = 2048 = 2^{11}$$

$$3^6 \cdot \mathbf{9} + 3^5 + 3^4 \cdot 2^2 + 3^3 \cdot 2^3 + 3^3 \cdot 2^4 + 3 \cdot 2^6 + 2^9 = = 2^{13}$$

$$3^4 \cdot \mathbf{11} + 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^3 + 2^6 = 2^{10}$$

$$3^2 \cdot \mathbf{13} + 3 + 2^3 = 2^7$$

$$3^5 \cdot \mathbf{15} + 3^4 + 3^3 \cdot 2 + 3^2 \cdot 2^2 + 3 \cdot 2^3 + 2^8 = 2^{12}$$

$$3^3 \cdot \mathbf{17} + 3^2 + 3 \cdot 2^2 + 2^5 = 2^9$$

$$3^6 \cdot \mathbf{19} + 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^4 + 3^3 \cdot 2^5 + 3 \cdot 2^7 + 2^{10} = = 2^{14}$$

$$3 \cdot \mathbf{21} + 1 = 64 = 2^6$$

$$3^4 \cdot \mathbf{23} + 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^2 + 2^7 = 2^{11}$$

$$3^7 \cdot \mathbf{25} + 3^6 + 3^5 \cdot 2^2 + 3^4 \cdot 2^3 + 3^3 \cdot 2^6 + 3^2 \cdot 2^7 + 3 \cdot 2^9 + 2^{12} = 2^{16}$$

$$3^{41} \cdot \mathbf{27} + 3^{40} + 3^{39} \cdot 2 + 3^{38} \cdot 2^3 + 3^{37} \cdot 2^4 + 3^{36} \cdot 2^5 + 3^{35} \cdot 2^6 + 3^{34} \cdot 2^7 + 3^{33} \cdot 2^9 + 3^{32} \cdot 2^{11} + 3^{31} \cdot 2^{12} + 3^{30} \cdot 2^{14} + 3^{29} \cdot 2^{15} + 3^{28} \cdot 2^{16} + 3^{27} \cdot 2^{18} + 3^{26} \cdot 2^{19} + 3^{25} \cdot 2^{20} + 3^{24} \cdot 2^{21} + 3^{23} \cdot 2^{23} + 3^{22} \cdot 2^{26} + 3^{21} \cdot 2^{27} + 3^{20} \cdot 2^{28} + 3^{19} \cdot 2^{30} + 3^{18} \cdot 2^{31} + 3^{17} \cdot 2^{33} + 3^{16} \cdot 2^{34} + 3^{15} \cdot 2^{35} + 3^{14} \cdot 2^{36} + 3^{13} \cdot 2^{37} + 3^{12} \cdot 2^{38} + 3^{11} \cdot 2^{41} + 3^{10} \cdot 2^{42} + 3^9 \cdot 2^{43} + 3^8 \cdot 2^{44} + 3^7 \cdot 2^{48} + 3^6 \cdot 2^{50} + 3^5 \cdot 2^{52} + 3^4 \cdot 2^{56} + 3^3 \cdot 2^{59} + 3^3 \cdot 2^{60} + 3^{32} \cdot 2^{11} + 3 \cdot 2^{61} + 2^{66} = 2^{70}$$

We can also write your formula (1) as follows:

$$2^k - 3^m \cdot n = 3^{m-1} \cdot 2^0 + 3^{m-2} \cdot 2^{k_{m-2}} + 3^{m-3} \cdot 2^{k_{m-3}} + + \cdots + 3^2 \cdot 2^{k_2} + 3^1 \cdot 2^{k_1} + 3^0 \cdot 2^{k_0} \quad (2)$$

B. *Theorem 2*

For each value of m the following identity holds:

$$3^m + 3^{m-1} \cdot 2^0 + 3^{m-2} \cdot 2^2 + 3^{m-3} \cdot 2^4 + 3^{m-4} \cdot 2^6 + \ldots + 3^1 \cdot 2^{2m-4} + 3^0 \cdot 2^{2m-2} = 2^{2m} \quad (3)$$

Below, the expansion of equation (2) is provided for m = 1, 2, and 3:

$$3^1 + 3^0 \cdot 2^0 = 3 + 1 \cdot 1 = 3 + 1 = 4 = 2^2 = 2^{2 \cdot 1}$$

$$3^2 + 3^1 \cdot 2^0 + 3^0 \cdot 2^2 = 9 + 3 \cdot 1 + 1 \cdot 4 = 9 + 3 + 4 = 16 = 2^4 = 2^{2 \cdot 2}$$

$$3^3 + 3^2 \cdot 2^0 + 3^1 \cdot 2^2 + 3^0 \cdot 2^4 = = 27 + 9 \cdot 1 + 3 \cdot 4 + 1 \cdot 16 = 27 + 9 + 12 + 16 = 64 = 2^6 = 2^{2 \cdot 3}$$

Using the formula (1), we can write the following theorem for n:

C. *Theorem 3*

$$n = \tfrac{1}{3}\left(\tfrac{1}{3}\left(\tfrac{1}{3} \ldots \tfrac{1}{3}(2^k - 2^{k_0}) - 2^{k_1}\right) - 2^{k_2}\right) - - \cdots - 2^{k_{m-3}}\Big) - 2^{k_{m-2}}\Big) - 1) \quad (4)$$

Here, m is the number of elements in CO(n) that are different from n, and the number of parentheses and "1/3"s is equal to m.

Below, the expansion of equation (4) for $n = 7$ is provided:

$$3^5 \cdot 7 + 3^4 + 3^3 \cdot 2 + 3^2 \cdot 2^2 + 3 \cdot 2^4 + 2^7 = 2^{11} \Rightarrow$$

$$3^4 \cdot 7 + 3^3 + 3^3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^4 = \frac{1}{3}(2^{11} - 2^7) \Rightarrow$$

$$3^3 \cdot 7 + 3^2 + 3 \cdot 2 + 2^2 = \frac{1}{3}\left(\frac{1}{3}(2^{11} - 2^7) - 2^4\right) \Rightarrow$$

$$3^3 \cdot 7 + 3^2 + 3 \cdot 2 + 2^2 = \frac{1}{3}\left(\frac{1}{3}(2^{11} - 2^7) - 2^4\right) \Rightarrow$$

$$3^2 \cdot 7 + 3 + 2 = \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}(2^{11} - 2^7) - 2^4\right) - 2^2\right) \Rightarrow$$

$$3 \cdot 7 + 1 = \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}(2^{11} - 2^7) - 2^4\right) - 2^2\right) - 2\right) \Rightarrow$$

$$3 \cdot 7 = \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}(2^{11} - 2^7) - 2^4\right) - 2^2\right) - 2\right) - 1 \Rightarrow$$

$$7 = \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}(2^{11} - 2^7) - 2^4\right) - 2^2\right) - 2\right) - 1\right) \Rightarrow$$

$$= \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}(2048 - 128) - 16\right) - 4\right) - 2\right) - 1\right) \Rightarrow$$

$$= \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3} \cdot 1920 - 16\right) - 4\right) - 2\right) - 1\right) \Rightarrow$$

$$= \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}(640 - 16) - 4\right) - 2\right) - 1\right) \Rightarrow$$

$$= \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3} \cdot 624 - 4\right) - 2\right) - 1\right) \Rightarrow$$

$$= \frac{1}{3}\left(\frac{1}{3}\left(\frac{1}{3} \cdot 204 - 2\right) - 1\right) \Rightarrow$$

$$= \frac{1}{3}\left(\frac{1}{3}(68 - 2) - 1\right) \Rightarrow$$

$$= \frac{1}{3}\left(\left(\frac{1}{3} \cdot 66\right) - 1\right) \Rightarrow$$

$$= \frac{1}{3}(22 - 1) \Rightarrow$$

$$= \frac{1}{3} \cdot 21$$

Since there are 5 elements different from "7" in CO(7), there are 5 "1/3"s and 5 parentheses in the expansion above for the number 7.

V. Conclusion

The Collatz Conjecture, although not yet fully solved mathematically, finds significant applications in fields such as computer science and cryptography. In this study, the mathematical foundation of the Collatz Conjecture and its various practical applications have been examined, and insights into future research directions have been provided. Future studies will further explore the potential of this conjecture in computer science, data security, and cryptography, and will pave the way for new approaches towards its mathematical solution.

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