Solvability Conditions of Two-Stage Transportation Problem with Two-Sided Constraints on Consumer Demands and Intermediate Points Capacities

Viktor Stovba
Department of Nonsmooth Optimization Methods
V.M. Glushkov Institute of Cybernetics of the NAS of Ukraine
Kyiv, Ukraine
vik.stovba@gmail.com

Petro Stetsyuk
Department of Nonsmooth Optimization Methods
V.M. Glushkov Institute of Cybernetics of the NAS of Ukraine
Kyiv, Ukraine
stetsyukp@gmail.com

***Abstract*—A mathematical model of a two-stage transportation problem with two-sided constraints on unknown consumer demands and intermediate points capacities is studied. Solvability conditions of the linear programming problem, which corresponds to this model, are formulated, and justified. The classical two-stage transportation problem is given, to which the above problem is transformed if consumer demands are known, and there are no lower and upper bounds on intermediate points capacities.**

Keywords—two-stage transportation problem, linear programming problem, two-sided constraints, solvability conditions

I. Introduction

Transportation type problems are highly popular nowadays, since they can be used to model a number of different situations, when some homogeneous product is to be transported from one set of points (usually called suppliers) to another (usually called consumers). Such a rather broad problem statement allows one to build a lot of modifications and constraints to the problem. Two-stage transportation problem (TTP) [1], [2] is a vivid example of such modifications, where products are transported from suppliers to consumers through a certain set of intermediate points, which can be represented by intermediary firms, warehouses, etc. For this problem, we can set upper limits on bandwidth of intermediate points [3], make it possible to choose a fixed number of intermediate points smaller than their total number [3], and set the upper limit of this quantity [4]. Studied are applications of TTP to another kind of problems and models as well, in particular, to the optimal set partitioning problem [5], p-median of a graph [3], rational location of a given number of storages [3] etc.

For different modifications of two-stage transportation problem necessary and sufficient conditions of consistency of constraint system of the problem are of great interest. For TTP with upper bounds on capacities of a fixed number of intermediate points the conditions are proposed and justified in [3]. For a two-stage transportation problem with lower and upper bounds on unknown consumer demands such conditions were formulated in [6] and justified in [7]. Finally, for a two-stage transportation problem with two-sided bounds on consumer demands and upper bounds on intermediate points capacities such results are given in [8]. The purpose of the article is to formulate and justify the necessary and sufficient conditions of consistency of the constraints system of the most general TTP modification (i.e. solvability conditions), where two-sided constraints are set both on intermediate points and unknown consumer demands.

II. Two-stage Transportation Problem with Two-Sided Constraints

Let m supply points $A_1, \ldots, A_m$ contain $a_1, \ldots, a_m$ units of products that are to be transported to n consumers $B_1, \ldots, B_n$ through l intermediate points $D_1, \ldots, D_l$. It is necessary to find an optimal product transportation plan, where c_{ik} is the cost of transporting a unit of product from the supplier A_i to the intermediate point D_k, and c_{kj} is the cost of transporting a unit of product from the intermediate point D_k to the consumer B_j, and consumer needs corresponding to the optimal plan. This problem is called a classical two-stage transportation problem:

$$f_{xy}^* = f(x^*, y^*) = \min_{x,y}\{f(x,y) =$$

$$= \sum_{i=1}^{m}\sum_{k=1}^{l} c_{ik}x_{ik} + \sum_{k=1}^{l}\sum_{j=1}^{n} c_{kj}y_{kj}\} \qquad (1)$$

subject to

$$\sum_{k=1}^{l} x_{ik} = a_i, \ \ i = \overline{1,m}, \qquad (2)$$

$$\sum_{k=1}^{l} y_{kj} = b_j, \ \ j = \overline{1,n}, \qquad (3)$$

$$\sum_{i=1}^{m} x_{ik} - \sum_{j=1}^{n} y_{kj} = 0, \ \ k = \overline{1,l}, \qquad (4)$$

$$x_{ik} \ge 0, \ \ y_{kj} \ge 0, \ \ i = \overline{1,m}, \ \ k = \overline{1,l}, \ \ j = \overline{1,n}. \qquad (5)$$

Here $x = \{x_{ik}\}_{i=\overline{1,m}}^{k=\overline{1,l}}$, where x_{ik} is the number of product units that are transported from the supplier A_i to the intermediate point D_k; $y = \{y_{kj}\}_{k=\overline{1,l}}^{j=\overline{1,n}}$, where y_{kj} is the number of product units that is transported from the intermediate point D_k to the consumer B_j. Consistency conditions of constraint system of the problem are well known and justified, for instance in [4].

Let us consider the problem (1) – (5) with two-sided constraints on intermediate points and unknown consumer demands. It has the following form: to find

$$f_{xyz}^* = f(x^*, y^*, z^*) = \min_{x,y}\{f(x,y,z) =$$

$$= \sum_{i=1}^{m}\sum_{k=1}^{l} c_{ik}x_{ik} + \sum_{k=1}^{l}\sum_{j=1}^{n} c_{kj}y_{kj}\} \qquad (6)$$

subject to

$$\sum_{k=1}^{l} x_{ik} = a_i,\ \ i = \overline{1,m}, \tag{7}$$

$$\sum_{k=1}^{l} y_{kj} = z_j,\ \ j = \overline{1,n}, \tag{8}$$

$$\sum_{i=1}^{m} x_{ik} - \sum_{j=1}^{n} y_{kj} = 0,\ \ k = \overline{1,l}, \tag{9}$$

$$d_k^{low} \le \sum_{i=1}^{m} x_{ik} \le d_k^{up},\ \ k = \overline{1,l}, \tag{10}$$

$$\sum_{j=1}^{n} z_j = \sum_{i=1}^{m} a_i, \tag{11}$$

$$b_j^{low} \le z_j \le b_j^{up},\ \ j = \overline{1,n}. \tag{12}$$

$$x_{ik} \ge 0,\ \ y_{kj} \ge 0,\ \ \ i = \overline{1,m},\ \ k = \overline{1,l},\ \ j = \overline{1,n}. \tag{13}$$

Here $d_1^{low}, \dots, d_l^{low}$ and $d_1^{up}, \dots, d_l^{up}$ are given lower and upper bounds on intermediate points capacities respectively. The volumes of consumer demands are considered unknown, and their lower bounds $b_1^{low}, \dots, b_n^{low}$ and upper bounds $b_1^{up}, \dots, b_n^{up}$ are given. It is necessary to find an optimal product transportation plan, where c_{ik} is the cost of transporting a unit of product from the supplier A_i to the intermediate point D_k, and c_{kj} is the cost of transporting a unit of product from the intermediate point D_k to the consumer B_j, and consumer needs corresponding to the optimal plan.

Further we will assume that $m \ge 2, n \ge 2, l \ge 2$, as well as $a_i > 0$, $b_j^{up} \ge b_j^{low} > 0$, $d_k^{up} \ge d_k^{low} > 0$, $i = \overline{1,m}$, $j = \overline{1,n}$. Otherwise, the problem is significantly simplified, and, as follows from constraints (7) – (13), a certain part of the variables x_{ik}, y_{kj}, z_j, $i = \overline{1,m}$, $k = \overline{1,l}$, $j = \overline{1,n}$ become zero and can be removed from further consideration.

Problem (6) – (13) is a linear programming problem (LP problem), which contains $m \times l + n \times (l+1)$ continuous variables x, y, z and $3(n+l) + m + 1$ linear constraints. The objective function $f(x, y, z)$ specifies the total cost of transporting products from suppliers to consumers. Constraints (7) denote transportation of $a_1, \dots, a_m$ units of production from suppliers to intermediate points, and constraint (8) means that consumers need to deliver unknown volumes $z_1, \dots, z_n$ of production units from intermediate points. Constraints (9) set the conditions for all products arriving from suppliers to each intermediate point to be necessarily sent to consumers. Constraints (10) set lower and upper bounds on intermediate points capacities. Constraint (11) sets the condition that the total output of suppliers equals the total output of consumers. Constraints (12) set lower and upper bounds on unknown consumer demands.

Problem (6) – (13) belongs to the balanced problems of transport type, that is, all products of suppliers must be delivered to consumers, leaving nothing in intermediate points. This dictates the conditions for consistency of the constraints system (7) – (13). The following theorem is valid.

Theorem. *Constraint system (6) – (13) is consistent if and only if the following conditions are satisfied:*

$$\sum_{j=1}^{n} b_j^{low} \le \sum_{i=1}^{m} a_i \le \sum_{j=1}^{n} b_j^{up}, \tag{14α}$$

$$\sum_{k=1}^{l} d_k^{low} \le \sum_{i=1}^{m} a_i \le \sum_{k=1}^{l} d_k^{up}. \tag{14β}$$

Proof. Necessity. Let there exist nonnegative $\bar{x} = \{\bar{x}_{ik},\ i = \overline{1,m}, k = \overline{1,l}\}$, $\bar{y} = \{\bar{y}_{kj},\ k = \overline{1,l}, j = \overline{1,n}\}$ and positive $\bar{z} = \{\bar{z}_j,\ j = \overline{1,n}\}$, which satisfy the constraint system (7) – (13). That is, for $\bar{x} \ge 0$, $\bar{y} \ge 0$ and $\bar{z} > 0$ the following equalities are satisfied:

$$\sum_{k=1}^{l} \bar{x}_{ik} = a_i,\ \ i = \overline{1,m}, \tag{15}$$

$$\sum_{k=1}^{l} \bar{y}_{kj} = \bar{z}_j,\ \ j = \overline{1,n}, \tag{16}$$

$$\sum_{i=1}^{m} \bar{x}_{ik} - \sum_{j=1}^{n} \bar{y}_{kj} = 0,\ \ k = \overline{1,l}, \tag{17}$$

$$d_k^{low} \le \sum_{i=1}^{m} \bar{x}_{ik} \le d_k^{up},\ \ k = \overline{1,l}, \tag{18}$$

$$\sum_{j=1}^{n} \bar{z}_j = \sum_{i=1}^{m} a_i, \tag{19}$$

$$b_j^{low} \le \bar{z}_j \le b_j^{up},\ \ j = \overline{1,n}. \tag{20}$$

After summing (20) by index $j = \overline{1,n}$ and using (19), we get

$$\sum_{j=1}^{n} b_j^{low} \le \sum_{j=1}^{n} \bar{z}_j = \sum_{i=1}^{m} a_i \le \sum_{j=1}^{n} b_j^{up},$$

which proves the validity of (14α). Also, summing up inequalities (18) by index $k = \overline{1,l}$ we get

$$\sum_{k=1}^{l} d_k^{low} \le \sum_{k=1}^{l} \sum_{i=1}^{m} \bar{x}_{ik} \le \sum_{k=1}^{l} d_k^{up}, \tag{21}$$

and, summing equalities (15) by index $i = \overline{1,m}$, we have

$$\sum_{i=1}^{m} \sum_{k=1}^{l} \bar{x}_{ik} = \sum_{i=1}^{m} a_i. \tag{22}$$

Changing order of summing of $\bar{x}_{ik}$ and using equality (22) for inequality (21) we obtain validity of (14β).

Sufficiency. Let (14α) and (14β) be fulfilled. We determine the index j^* in the following way:

$$\sum_{j=1}^{j^*} b_j^{up} + \sum_{j=j^*+1}^{n} b_j^{low} \le \sum_{i=1}^{m} a_i, \tag{23α}$$

$$\sum_{j=1}^{j^*+1} b_j^{up} + \sum_{j=j^*+2}^{n} b_j^{low} > \sum_{i=1}^{m} a_i. \tag{23β}$$

Index k^* is determined in the same way, but for values b_j^{low} and b_j^{up}, i.e.

$$\sum_{k=1}^{k^*} d_k^{up} + \sum_{k=k^*+1}^{l} d_k^{low} \le \sum_{i=1}^{m} a_i, \tag{24α}$$

$$\sum_{k=1}^{k^*+1} d_k^{up} + \sum_{k=k^*+2}^{l} d_k^{low} > \sum_{i=1}^{m} a_i. \tag{24β}$$

Let us define vector $b = \left(b_1^{up}, \dots, b_{j^*}^{up}, t_1, b_{j^*+2}^{low}, \dots, b_n^{low}\right) \in R^n$, where $t_1 = \sum_{i=1}^{m} a_i - \sum_{j=1}^{j^*} b_j^{up} - \sum_{j=j^*+2}^{n} b_j^{low}$, and vector $d = \left(d_1^{up}, \dots, d_{k^*}^{up}, t_2, d_{k^*+2}^{low}, \dots, d_l^{low}\right) \in R^l$, where $t_2 = \sum_{i=1}^{m} a_i - \sum_{k=1}^{k^*} d_k^{up} - \sum_{k=k^*+2}^{l} d_k^{low}$. Let us consider variables x_{ik} and y_{kj}, $i = \overline{1,m}$, $k = \overline{1,l}$, $j = \overline{1,n}$, built in the following way: $\{x_{ik}\}_{i=\overline{1,m}}^{k=\overline{1,l}} = \left\{\frac{d_k a_i}{\sum_{i=1}^{m} a_i}\right\}_{i=\overline{1,m}}^{k=\overline{1,l}}$, $\{y_{kj}\}_{k=\overline{1,l}}^{j=\overline{1,n}} = \left\{\frac{d_k b_j}{\sum_{i=1}^{m} a_i}\right\}_{j=\overline{1,n}}^{k=\overline{1,l}}$. Variable $z = \{z_j\}_{j=\overline{1,n}}$ equals vector b. Let us show that variables x_{ik}, y_{kj} and z_j, $i=\overline{1,m}$, $k=\overline{1,l}$, $j=\overline{1,n}$, satisfy constraints (7) – (13).

Constraint (7):

$$\sum_{k=1}^{l} x_{ik} = \sum_{k=1}^{l} \frac{d_k a_i}{\sum_{i=1}^{m} a_i} = \frac{a_i}{\sum_{i=1}^{m} a_i}\sum_{k=1}^{l} d_k =$$
$$= \frac{a_i}{\sum_{i=1}^{m} a_i}\sum_{i=1}^{m} a_i = a_i,\ i=\overline{1,m},$$

since

$$\sum_{k=1}^{l} d_k = \sum_{k=1}^{k^*} d_k^{up} + \sum_{i=1}^{m} a_i - \sum_{k=1}^{k^*} d_k^{up} -$$
$$- \sum_{k=k^*+2}^{l} d_k^{low} + \sum_{k=k^*+2}^{l} d_k^{low} = \sum_{i=1}^{m} a_i. \qquad (25)$$

Constraint (8):

$$\sum_{k=1}^{l} y_{kj} = \sum_{k=1}^{l} \frac{d_k b_j}{\sum_{i=1}^{m} a_i} = \frac{b_j}{\sum_{i=1}^{m} a_i}\sum_{k=1}^{l} d_k =$$
$$= \frac{b_j}{\sum_{i=1}^{m} a_i}\sum_{i=1}^{m} a_i = b_j = z_j,\ j=\overline{1,n}.$$

Constraint (9):

$$\sum_{i=1}^{m} x_{ik} - \sum_{j=1}^{n} y_{kj} = \sum_{i=1}^{m} \frac{d_k a_i}{\sum_{i=1}^{m} a_i} - \sum_{j=1}^{n} \frac{d_k b_j}{\sum_{i=1}^{m} a_i} =$$
$$= \frac{d_k}{\sum_{i=1}^{m} a_i}\sum_{i=1}^{m} a_i - \frac{d_k}{\sum_{i=1}^{m} a_i}\sum_{j=1}^{n} b_j =$$
$$= \frac{d_k}{\sum_{i=1}^{m} a_i}\sum_{i=1}^{m} a_i - \frac{d_k}{\sum_{i=1}^{m} a_i}\sum_{i=1}^{m} a_i = 0,\ k=\overline{1,l},$$

since

$$\sum_{j=1}^{n} b_j = \sum_{j=1}^{j^*} b_j^{up} + \sum_{i=1}^{m} a_i - \sum_{j=1}^{j^*} b_j^{up} -$$
$$- \sum_{j=j^*+2}^{n} b_j^{low} + \sum_{j=j^*+2}^{n} b_j^{low} = \sum_{i=1}^{m} a_i. \qquad (26)$$

Constraint (10): using inequality (14b), we get

$$\sum_{i=1}^{m} x_{ik} = \sum_{i=1}^{m} \frac{d_k a_i}{\sum_{i=1}^{m} a_i} = \frac{d_k}{\sum_{i=1}^{m} a_i}\sum_{i=1}^{m} a_i = d_k,\ k=\overline{1,l}.$$

Now let us show that $d_k^{low} \le d_k \le d_k^{up}$, $k=\overline{1,l}$. If $k=\overline{1,k^*}$ we have $d_k^{low} \le d_k^{up} \le d_k^{up}$, and if $k=\overline{k^*+2,l}$ we have $d_k^{low} \le d_k^{low} \le d_k^{up}$. Consider case with $k=k^*+1$. Using (24β), we get

$$d_{k^*+1} = t_2 = \sum_{i=1}^{m} a_i - \sum_{k=1}^{k^*} d_k^{up} - \sum_{k=k^*+2}^{l} d_k^{low} <$$
$$< \sum_{k=1}^{k^*+1} d_k^{up} + \sum_{k=k^*+2}^{l} d_k^{low} - \sum_{k=1}^{k^*} d_k^{up} -$$
$$- \sum_{k=k^*+2}^{l} d_k^{low} = d_{k^*+1}^{up},$$

and using (24α) we obtain

$$d_{k^*+1} = t_2 = \sum_{i=1}^{m} a_i - \sum_{k=1}^{k^*} d_k^{up} - \sum_{k=k^*+2}^{l} d_k^{low} =$$
$$= d_{k^*+1}^{low} + \sum_{i=1}^{m} a_i - \sum_{k=1}^{k^*} d_k^{up} - d_{k^*+1}^{low} -$$
$$- \sum_{k=k^*+2}^{l} d_k^{low} = d_{k^*+1}^{low} + \sum_{i=1}^{m} a_i - \sum_{k=1}^{k^*} d_k^{up} -$$
$$- \sum_{k=k^*+1}^{l} d_k^{low} \ge d_{k^*+1}^{low}.$$

Constraint (11) is valid since $\{z_j\}_{j=\overline{1,n}} = \{b_j\}_{j=\overline{1,n}}$ and $\sum_{j=1}^{n} b_j = \sum_{i=1}^{m} a_i$ (equality (26)).

Constraint (12). Since $\{z_j\}_{j=\overline{1,n}} = \{b_j\}_{j=\overline{1,n}}$ and vector b is built similarly to vector d, validity of constraint (12) can be shown in the same way as it is done for constraint (10), using inequalities (23α) and (23β).

Constraint (13). Variables x_{ik} and y_{kj} are non-negative, and z_j is positive by construction, since $b_j^{up} \ge b_j^{low} > 0$ ($j=\overline{1,n}$), $d_k^{up} \ge d_k^{low} > 0$ ($k=\overline{1,l}$), and $a_i \ge 0$ ($i=\overline{1,m}$). Moreover, at least one variable a_i is positive, otherwise the volume of products to be transported is zero.

Theorem is proved.

III. Conclusion

Mathematical model of linear programming for a two-stage transportation problem with two-sided constraints on unknown consumer demands and intermediate points capacities is considered. Solvability conditions of the problem are formulated and justified, which can be used in software implementations of algorithms for solving various modifications of two-stage transportation problems. Such modifications can be successfully used for numerous applied problems, in particular, for the problem of electric power storages placement in the IPS of Ukraine considering its influence on power flows transmitted by controlled cutsets [9].

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